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A Complete Characterization of Equilibria in Common Agency Screening Games [*]

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Abstract

We characterize the complete set of equilibrium allocations to intrinsic common agency screening games as the set of solutions to self-generating optimization programs. This analysis is performed both for continuous and discrete two-type models. These programs, in turn, can be thought of as maximization problems faced by a fictional surrogate principal with a simple set of incentive constraints that embed the non-cooperative behavior of principals in the underlying game. For the case of continuous types, we provide a complete characterization of equilibrium outcomes for regular environments by relying on techniques developed elsewhere for aggregate games and mechanism design problems with delegation. Those equilibria may be non-differentiable and/or exhibit discontinuities. Among those allocations, we stress the role the maximal equilibrium exhibits a n-fold distortion due to the principals' non-cooperative behavior. It is the unique equilibrium which is implemented by a tariff satisfying a biconjugacy requirement inherited from duality in convex analysis. This maximal equilibrium may not be the most preferred equilibrium allocation from the principals' point of view. We perform a similar analysis in the case of a discrete two-type model. We select within a large set of equilibria by imposing the same requirement of biconjugacy on equilibrium tariffs. Those outcomes are limits of equilibria exhibiting much bunching in nearby continuous type models which fail to be regular and require the use of ironing procedures.

Key words: Intrinsic common agency, aggregate games, mechanism design with delegation, duality, ironing procedures.

JEL Classification: D82, D86.

1. INTRODUCTION

Motivation and Objectives. We consider a canonical class of common agency games in which the agent has private information, his action is publicly contractable by all principals, and he must either accept all contract offers from the principals or choose not to participate. Common agency is thus *public* and *intrinsic.*[1] As a motivating example, suppose there are multiple government agencies (principals) who regulate a polluting public utility (the common agent) which has private information about the cost of production. If the firm decides to produce, it is under the joint control of all regulators. Regulators, however, may have conflicting objectives. For example, an environmental agency wishes on the margin to reduce output; a public-utility commission instead prefers to increase output and consumer surplus. Regulators simultaneously offer menus of transfer-output pairs in order to influence the choice of the public utility.

One of the main theoretical difficulties when modeling non-cooperative scenarios is to characterize the multitude of equilibrium outcomes that can arise. In more familiar single-principal screening environments, the *Revelation Principle* defines the set of relevant communication strategies and describes feasible allocations by means of incentive compatibility constraints.[2] With multiple principals, however, the *Revelation Principle* is neither simple to apply nor particularly useful. Even though the *Delegation Principle* proposed in Martimort and Stole (2002)[3] does offer a simple and universal representation of the strategy spaces available to mechanism designers in common agency environments, this tool fails to give a complete representation of equilibrium allocations. As a result, the literature on common agency has primarily focused on specific equilibria in structured games rather than exploring the entire set of equilibrium possibilities. To illustrate, authors have often analyzed differentiable equilibria in models with a continuum of types both because of their tractability and because economic insights drawn from these studies appear attractive. The arbitrariness of such a selection may nevertheless raise some concerns. Comparative statics and economic implications could be fragile. In particular, the welfare cost of the principals's non-cooperative behavior could be overestimated. A more complete approach - the tack of the present paper - is to characterize the entire set of equilibria in order to understand the full import of particular equilibrium refinements.

The goal of this paper is thus to make progress on three fronts: equilibrium characterization, equilibrium selection, and welfare comparisons. Readers should keep in mind that, our analysis aim at providing tool-kits for researchers willing to undertake more applied works in structured environments. To this end, we thus first characterize the complete set of equilibria of a canonical intrinsic common agency game where the agent's private information takes values drawn from a continuous distribution; a modeling framework that has been an important vehicle for screening models in the existing literature. Second, we perform the same exercise when the agent's type can only take two possible values. Although specific, this two-type screening model is also widely adopted in applications.[4] We then draw a natural link between those two models so as to sharpen predictions.

[1] See Martimort (2007) and Martimort and Stole (2016) for definitions.

[2] Myerson (1982).

[3] Also sometimes referred to as *Menus Theorems* in the parlance of Peters (2001).

[4] Two-type screening models are also the only vehicles to describe more complex environments with collusion or different sorts of imperfect commitment. Introducing such contractual imperfections in common agency environments requires discrete models.

Tools from Aggregate Games. Our first step towards a full characterization of equilibria relies on the fundamental structure of intrinsic common agency games. As noted by Martimort and Stole (2012), those games are a special case of *aggregate games:* Principal i's expected payoff depends only upon principal i's contract and the *aggregate contract* offered by all non-cooperating principals. Because of this feature, each principal can always *undo* the aggregate contract offered by others with his own tariff so as to implement whatever incentive-feasible allocation he would like.[5] However, in equilibrium, all principals must agree on inducing the same allocation. Everything happens as if they were to delegate the choice of such an allocation to *a surrogate principal.* This *Aggregate Concurrence Principle*, as coined by Martimort and Stole (2012), is a key ingredient to characterize the set of *all* equilibrium outcomes as solutions to simple *self-generating* optimization problems (Proposition 1).

Self-Generating problems. Such problems are *self-generating* in the sense that each solution maximizes an objective function that depends on an aggregate contract which, in turn, implements the solution. Thus, a solution appears both in the maximand and as a maximizer, generating a fixed point. This approach respects thus the fixed-point nature of equilibrium but it also imports much of the tractability and techniques found in solving simpler optimization problems. From a technical viewpoint, the fact that there is a single optimization problem summarizing equilibrium behavior (and not a pile of n different optimization problems, one for each principal) allows us to derive important properties of the surrogate principal's value function (e.g., absolute continuity, envelope condition) that help characterizing equilibrium output.

Although *self-generating* maximization problems look like the cooperative problem that n principals would face if they were able to collude in the design of the agent's contract, a key difference is that, in the former, the agent's rent is weighted n times more than in the latter. This n-fold excess weighting captures the fact that, in the non-cooperative scenario, each principal attempts to extract the agent's information rent for himself. Beyond this difference with the cooperative design, the two problems are otherwise similar. The surrogate principal maximizes the *self-generating* objective with a bias towards *overharvesting* the agent's information rent.

Equilibrium Sets. Martimort and Stole (2012) have used the force of these self-generating problems to prove equilibrium existence in intrinsic common agency games under quite general conditions (general type spaces, action sets, and preferences). This paper goes beyond existence and describes all equilibrium allocations in structured environments. To get sharp predictions, we assume that the agent's preferences is bilinear in output and type. This allows us to import powerful tools from convex analysis and duality at minimal cost for exposition. In particular, our first result (Theorem 1) characterizes the whole set of equilibrium allocations for a continuum of types by means of a new set of incentive constraints that characterizes the *surrogate principal*'s optimization behavior.

Tools from Mechanism Design. The existing large set equilibrium allocations can be

[5] This trick was first explained by Bernheim and Whinston (1986a) in a moral hazard setting, but it applies in our screening model as well.

best understood in viewing the principals' choice of a set of equilibrium outputs as a *delegation problem.* Indeed, everything happens as if non-cooperating principals were jointly delegating to their surrogate representative the decision to choose output for each possible realization of the agent's type. Of course, the difference in objectives between the principals acting collectively and their surrogate captures the loss due to their non-cooperative behavior. Importantly, this *delegation problem* can be studied using techniques that were recently developed in the mechanism design literature. (Holmström (1984), Melumad and Shibano (1991), Martimort and Semenov (2006), Alonso and Matouschek (2008), and Amador and Bagwell (2013) among others.) This literature has shown how the delegation problem can be solved with allocations that are either independent of the privately informed party's information or that instead correspond to his ideal point. This important insight carries over into our setting. At points where the equilibrium output is differentiable and separating, it can be identified with the pointwise maximizer of the *surrogate surplus* (Proposition 2); the so called *maximal* allocation. Elsewhere, equilibrium allocations might entail bunching and be non-responsive to the agent's private information over some range (Proposition 3).

Maximal Equilibrium. In *regular environments* characterized by the monotonicity of the hazard rate of the types distribution, the maximal allocation is itself an equilibrium that features a n-fold asymmetric information distortion. This allocation is remarkable for at least two reasons. First, it has been the focus of all applied research in public screening environments to date although existing works (Laffont and Tirole (1993, Chapter 17), Martimort and Semenov (2008), Martimort and Stole (2009a, 2009b) among others) have neglected the existence and characterization of other possible equilibria. Second, this allocation is *maximal* in the sense that this is the only one that can be implemented with a tariff satisfying a simple *biconjuguacy* property inherited from convex analysis. Binconjugate tariffs are the least-concave contracts implementing a given allocation. Those tariffs stand in contrast with forcing contracts. Forcing contracts help to prevent deviations by means of infinite punishments off path and thus sustain more equilibrium outcomes. Imposing biconjugacy acts thus as a refinement of the equilibrium set.

Other equilibria can nevertheless be constructed from the maximal allocation by introducing gaps in the range of equilibrium outputs. These allocations exhibit bunching, even in regular environments satisfying the monotone hazard rate condition, and discontinuities at points where the *surrogate surplus* nevertheless remains constant. Tariffs in those equilibria entail large negative payments over the discontinuity gaps. These punishments prevent not only the agent to choose outputs in the gap but also principals to deviate by proposing contracts that would push the agent to chose outputs there. Of course, these tariffs are no longer biconjugate.

Equilibrium selection. Although attractive because of the above inclusion property and the fact that it is the unique *biconjugate* equilibrium with a continuum of types (Proposition 4), the *maximal* equilibrium might not be the best equilibrium from the principals' *ex ante* point of view. Other more collusive equilibrium outcomes can be obtained if principals restrict the choice of output so as to avoid the excessive rent extraction that comes with this maximal allocation. Making again use of recent advances in the delegation literature (Amador and Bagwell (2013)), we demonstrate that, under mild conditions on the types distribution, the best equilibrium is *never* the maximal one

(Proposition 5). The best equilibrium puts a *"floor"* on outputs (by means of infinite punishments for outputs below this floor) so as to prevent excessive rent extraction.

A two-type Discrete Model. Because one of our goals if to offer ready-to-use toolkits for applied works, we also provide characterization results in the simpler case where the agent's type is discrete, an information structure of particular interest for modelers. Of course, the techniques of self-generation remains available in this setting. The solution to the self-generating problems are again characterized by a set of incentive constraints that describe the surrogate principal's behavior. Now, those constraints are non-local thanks to the discrete nature of the types distribution (Theorem 2). There again exists a large class of equilibria that can be implemented with forcing contracts.

Biconjugate Equilibria. While we offer a complete characterization of all equilibria in our two-type setting, we are also interested in the subset of equilibria which are analogues to the *maximal* equilibria of games with continuous types. Modelers may expect that those equilibria exhibit the same comparative statics than the differentiable equilibrium they are used to select in continuous types model. The appropriate analogue is obtained when the aggregate transfer offered by the principals again satisfies the *biconjugacy* property. Unsurprisingly, the qualitative predictions obtained with *biconjugate* equilibria in our two-type setting replicate the properties of the *maximal* equilibrium of continuous type games (Theorem 3). *Biconjugate* equilibria feature outputs distortions that are reminiscent of the familiar n-fold marginalization found for the *maximal* equilibrium of continuous type models.

Welfare. The best *biconjugate* equilibrium from the principals' point of view is as close as possible to the cooperative outcome. Yet, this solution is always dominated by an equilibrium implemented with forcing contracts (Propositions 6 and 7); a result that echoes our previous findings for continuous types. In other words, insisting on *biconjugate* equilibrium allocations entails a welfare loss for the principals. Better equilibrium outcomes are possible if principals rely on forcing contracts.

From Continuous to Discrete Types Models. Given the negative welfare result found above, other arguments must be made to justify using *biconjugate* equilibria in discrete environments. One possible rationale is to view discrete models as the limits of a continuous models where types are drawn from distributions which would converge towards a discrete two-type distribution, hoping that such convergence would select *biconjugate* allocations. Unfortunately, along this convergence, distributions can no longer satisfy the monotonicity of the hazard rate assumption. Self-generating maximization problems are not regular and bunching is pervasive for their solutions as one comes closer to the discrete type scenario. In particular, the *maximal* allocation is no longer non-decreasing. We borrow from Myerson (1981)'s seminal analysis, the *ironing* techniques that are necessary to describe solutions to the *self-generating* problems when regularity fails. The important contribution of Myerson (1981)'s approach, which is based on a transformation of the agent's virtual cost to preserve monotonicity of the solution, is that it does not impose smoothness conditions (like for instance continuity or differentiability) that might be necessary to use optimal control to solve for best responses.[6] Indeed, such conditions should

[6]Toikka (2011) makes a similar point.

not be imposed *a priori* if one wants to have a full account of equilibrium allocations. Equipped with this tool, we show that sequences of equilibrium allocations of nearby continuous type models converge towards a *biconjugate* equilibrium of the discrete model (Theorem 4) under weak conditions on distributions in these nearby models. This limit result justifies focusing on biconjugate equilibria in two-type models.

Literature review. Existing characterization results for common agency models are quite fragmented and cover various contracting scenarios. Assuming symmetric information and *delegated common agency* with public contracts, Bernheim and Whinston (1986a) and Laussel and Lebreton (1998, 2001) have described payoffs for the so-called *truthful equilibria* while Chiesa and Denicolò (2009) have investigated the case of private contracts. The former authors focus on *truthful tariffs* (maybe because they are coalition-proof as proved in Bernheim and Whinston (1986a)) to ensure that each principal's contribution exactly reflects his preferences among alternatives. Efficiency follows. The only remaining question is how the possibility that the agent may reject some offer redistributes surplus among players.

Under asymmetric information,the distributions of equilibrium payoffs can no longer be disentangled from the allocative distortions that arise at equilibrium. Martimort and Stole (2015) present necessary conditions which are satisfied by all equilibrium outcomes of delegated public common agency games when the agent has private information drawn from a continuous types distribution. They derive *maximal equilibria* in those contexts and observe that they differ from *maximal equilibria* in intrinsic games because tariffs must remain non-negative under delegated agency. These necessary conditions remain compact enough to describe both continuous and discontinuous equilibrium allocations. Yet, sufficiency has to be checked on a case-by-case basis. This is in contrast with the analysis of intrinsic games developed hereafter where sufficiency is immediate. The difference comes from the fact that intrinsic games are *bijective* aggregate games in the parlance of Martimort and Stole (2012). Knowing the solution to the self-generating problem is enough to recover solutions to all principals' optimization problems. Delegated agency games do not satisfy bijectivity since the possibility of rejecting any offer implies that contracts are necessarily non-negative.

Among others, Stole (1991), Martimort (1992), Martimort and Stole (2009a), Calzolari (2001) and Calzolari and Denicolò (2013) for private contracting, Laffont and Tirole (1993, Chapter 17), Laussel and Lebreton (1998), Martimort and Semenov (2008), Martimort and Stole (2009b), Hoernig and Valletti (2011) for public contracting have described various differentiable equilibria that arise under asymmetric information in intrinsic common agency games with a continuum of types. None of these papers investigate the full set of equilibria as we do thereafter. This step is possible by building on techniques similar to those Martimort and Stole (2015) but now specialized to intrinsic common agency games. Laussel and Resende (2016) also tackle this problem in the specific context of competing manufacturers. Beside other technical differences, their approach in characterizing equilibrium allocations proceeds by deriving necessary conditions based on individual best responses which are stricter than ours. This leaves aside the issue of whether the allocations so found are indeed equilibria. Necessary and sufficient conditions are obtained altogether with our approach based on viewing equilibria as solutions to self-generating problems for bijective aggregate common agency games. Moreover, our approach allows us to directly identify equilibrium output profiles with implementable allocations of a simple mechanism design problem of delegation and import the findings of this literature to qualify Pareto-dominant equilibria.

Turning to the case of private and delegated common agency, Biais et al. (2000) have studied competition in convex nonlinear tariffs for common values models that satisfy monotonicity of the hazard rate. In contrast, discrete types models have featured inexistence of equilibria as shown in Attar et al. (2014a, 2014b) who do not restrict attention to convex equilibria nor to any particular class of preferences for the agent, beyond single crossing. On a different ground, the convergence results in the present paper contribute to understanding the differences between discrete and continuous models.

Organization. Section 2 presents the model. Section 3 describes the set of incentive feasible allocations for continuous types. We present there some of the duality tools that are used throughout the paper, defining in particular the notion of *biconjugacy*. We also briefly review the cooperative benchmark. Section 4 presents the *self-generating* optimization problems that represent equilibria. Section 5 characterizes those equilibria. Section 6 discusses equilibrium selection and welfare. Section 7 tackles the case of discrete types. We again provide a full characterization of equilibrium allocations, finds the optimal one from the principals' point of view, and justifies the focus on *biconjugate* equilibria as limits of equilibria in nearby continuous type models. Proofs are relegated to an Appendix.

2. AN INTRINSIC COMMON AGENCY GAME

The focus of this paper is on common agency games with $n > 1$ principals (indexed by $i \in \{1, .., n\}$), each of whom contracts with a single common agent. We assume that common agency is *intrinsic* and the choice variable of the agent is *public* (i.e., commonly observable and contractible by all principals).

Preferences. Each principal i and the agent have preferences over output, $q \in \mathcal{Q}$, and payments that are, respectively, defined as

$$S_i(q) - t_i \text{ (principal } i) \quad \forall i \in \{1, .., n\} \text{ and } \sum_{i=1}^{n} t_i - \theta q \text{ (common agent)}.$$

We assume that the payoff functions S_i (for $i \in \{1, .., n\}$) are strictly concave and twice continuously differentiable. Without loss of generality, we normalize $S_i(0) = 0$, so $S_i(q)$ should be understood as the net utility of principal i relative to the non-contractual default, $q = 0$.[7] We denote aggregate gross surplus by $S(q) = \sum_{i=1}^{n} S_i(q)$ (with $S(0) = 0$) and by $S_{-i}(q) = \sum_{j \neq i} S_j(q)$ the aggregate gross surplus for all principals except i.

[7]That the agent's utility function is bilinear in θ and q allows us to import many direct results from duality theory from convex analysis (for instance, our notion of *biconjugacy* below). These findings could be generalized to preferences for the agent of the sort $t_i + u(\theta, q)$ for some u function. The relevant notions of convexity is *u-convexity* as discussed in Carlier (2001) and Basov (2005, Chapter 3). A specific case that is directly amenable to our analysis arises with some form of additivity. To illustrate, suppose that the agent's payoff is $\sum_{i=1}^{n} t_i + S_0(q) - \theta q$, where S_0 is a concave function normalized at $S_0(0) = 0$ that represents the agent's intrinsic benefit of production. Redefine payments from each principal and their respective payoff functions so that $\tilde{t}_i = t_i - \frac{S_0(q)}{n}$ and $\widetilde{S}_i(q) = S_i(q) + \frac{S_0(q)}{n}$ (for $i \in \{1, .., n\}$). One can verify that $\widetilde{S}_i(0) = 0$ and the expressions for the principals' and the agent's utility functions can be written, respectively, as $\tilde{S}_i(q) - \tilde{t}_i$ and $\sum_{i=1}^{n} \tilde{t}_i - \theta q$, just as we assume in the main text.

Contracts. From the *Delegation Principle*,[8] there is no loss of generality in studying pure-strategy common agency equilibria to require that principals' strategy spaces are restricted to tariffs from output to transfers. As such, we denote each principal's strategy space, $\mathcal{T}$, as the set of all upper semicontinuous mappings, T_i, from $\mathcal{Q}$ into $\mathbb{R}$ (for $i \in \{1, .., n\}$). We denote by $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{T}^n$ an arbitrary array of contracts.[9] An *aggregate contract* (or, in short, an *aggregate*) is defined as $T(q) = \sum_{i=1}^n T_i(q)$. We also use sometimes the familiar notation $\mathbf{T}_{-i}$ and $T_{-i}(q) = \sum_{j \neq i} T_j(q)$ to denote, respectively, an array of contracts and the aggregate contract from all principals but i.

Timing and Information. The timing is typical of principal-agent screening games, but now with n principals contracting instead of one. First, the agent privately learns his type (a cost parameter), θ that belongs to a set Θ. In the sequel and although we keep exposition as general as possible to cover more possibilities, we will entertain two cases. In the first one, the set Θ is an interval $[\theta_1, \theta_2]$ (with $\Delta\theta = \theta_2 - \theta_1$) and θ is distributed over this set according to a cumulative distribution $F(\theta)$ with an atomless and positive density $f(\theta)$. In the second scenario of interest especially in view of applied works, θ takes values in a discrete set $\Theta = \{\theta_1, \theta_2\}$, where $\theta = \theta_1$ (resp. $\theta = \theta_2$) with probability ν (resp. $1 - \nu$). Let $\mathrm{E}_\theta[\cdot]$ denote the expectation operator for the types distribution.

Second, principals simultaneously offer the agent the tariffs, $T_i : \mathcal{Q} \to \mathbb{R}$, which are promises to pay $T_i(q)$ to the agent following the choice of $q \in \mathcal{Q} = [0, q_{\max}]$. Our assumption that common agency is *public* is captured by the fact that all principals contract on the same observed choice by the agent.

Third, the agent either accepts or rejects all of the principals' offers. Refusing gives a reservation payoff that is normalized at zero. That common agency is *intrinsic* means that the agent must either accept or reject all contracts; partial participation is not an option. We denote the agent's acceptance decision by the strategy $\delta = 1$ and rejection by $\delta = 0$. If all contracts are accepted, the agent then chooses $q \in \mathcal{Q}$ to maximize his utility and receives payments from each principal according to their contractual offers. Upper semicontinuity together with the compactness of $\mathcal{Q}$ ensure that the agent will always find an optimal output. We assume that if the agent rejects the contracts, then by default all transfers are nil. Thus, the agent's strategy is a pair, $\{\delta, q\}$, depending upon the agent's type and the contracts offered by the principals. Observe that if the agent rejects all contracts ($\delta = 0$), he chooses $q = 0$ which yields zero payoff to all players.

Equilibrium: Our focus in this paper is on equilibrium allocations that arise in a pure-strategy Perfect Bayesian equilibrium.

Definition 1 *An equilibrium is a n+2-tuple* $\{\overline{T}_1, \ldots, \overline{T}_n, \overline{q}_0, \overline{\delta}_0\}$ *(with aggregate* $\overline{T}(q) = \sum_{i=1}^n \overline{T}_i(q)$*) such that*

[8] Martimort and Stole (2002).

[9] We do not consider stochastic payment schedules because they have no value in our context with risk neutral players. Any stochastic payment schedule that would offer a lottery over payments for a given value of the agent's output could be replaced by the corresponding expected payment without changing payoffs and incentives. Also, we do not consider the possibility of writing contracts on contracts as in Szentes (2015).

1. $\overline{q}_0(\theta, \mathbf{T})$ *and* $\overline{\delta}_0(\theta, \mathbf{T})$ *jointly maximize the agent's payoff:*

$$\{\overline{q}_0(\theta, \mathbf{T}), \overline{\delta}_0(\theta, \mathbf{T})\} \in \arg\max_{q \in \mathcal{Q}, \delta \in \{0,1\}} \delta T(q) - \theta q \quad \forall \theta \in \Theta, \forall T \in \mathcal{T}.$$

2. $\overline{T}_i$ *maximizes principal* i*'s expected payoff given the other principals' contracts* $\overline{\mathbf{T}}_{-i}$*:*

$$\overline{T}_i \in \arg\max_{T_i \in \mathcal{T}} E_\theta \left[S_i(\overline{q}_0(\theta, T_i, \overline{\mathbf{T}}_{-i})) - \overline{\delta}_0(\theta, T_i, \overline{\mathbf{T}}_{-i})) T_i(\overline{q}_0(\theta, T_i, \overline{\mathbf{T}}_{-i})) \right], \quad \forall \theta \in \Theta.$$

For any equilibrium, $\{\overline{\mathbf{T}}, \overline{q}_0, \overline{\delta}_0\}$, we define the associated equilibrium allocation as the triplet: $\overline{\delta}(\theta) = \overline{\delta}_0(\theta, \overline{\mathbf{T}})$, $\overline{q}(\theta) = \overline{q}_0(\theta, \overline{\mathbf{T}})$, and $\overline{U}(\theta) = \delta(\theta)\overline{T}(\overline{q}(\theta)) - \theta\overline{q}(\theta)$.

In what follows, it will be useful to refer to the set of type-allocation mappings that are implementable for some aggregate tariff, $\mathcal{I}$, and to the subset of those type-allocation mappings that arise in some equilibrium, $\mathcal{I}^{eq}$.

Definition 2 *A type-allocation mapping,* (U, q, δ)*,* $U : \Theta \to \mathbb{R}$*,* $q : \Theta \to \mathcal{Q}$*,* $\delta : \Theta \to \{0,1\}$*, is implementable if there is an aggregate tariff,* $T : \mathcal{Q} \to \mathbb{R}$ *such that*

$$(q(\theta), \delta(\theta)) \in \arg\max_{q \in \mathcal{Q}, \delta \in \{0,1\}} \delta T(q) - \theta q,$$

$$U(\theta) = \max_{q \in \mathcal{Q}, \delta \in \{0,1\}} \delta T(q) - \theta q.$$

The set of all implementable allocations is denoted $\mathcal{I}$*.*

A type-allocation mapping $(\overline{U}, \overline{q}, \overline{\delta})$ *is an equilibrium allocation, or equilibrium implementable, if it is implementable by an aggregate tariff,* $\overline{T}$*, that arises at equilibrium.*

Proposition 1 below shows that, for any equilibrium allocation $(\overline{U}, \overline{q}, \overline{\delta})$, there exists another equilibrium allocation in which the agent always participates, $\delta(\theta) = 1$ for all θ, and so we will subsequently focus our attention on the pair $(\overline{U}, \overline{q})$ and suppress the type-allocation mapping for δ. Yet, our more general formulation remains useful for two reasons. First, it allows us to incorporate the agent's decision to participate as a requirement of implementability.[10] Second, it accounts for the possibility of equilibria where principals make non-serious offers. Indeed, there always exist uninteresting, trivial equilibria induced by a coordination failures in which two or more principals require sufficiently negative payments for each $q \in \mathcal{Q}$ so that it is not profitable for any principal to induce agent participation and $\delta = 0$ for such equilibrium allocations.

Full Information Allocation. The first-best allocation (U^{fb}, q^{fb}) is obtained when principals cooperate and know the agent's cost parameter. In this scenario, principals jointly request production at the first-best level, $q^{fb}(\theta)$, and set transfers which jointly extract the agent's surplus. Assuming $S'(0) \geq \theta_2$ to avoid corner solution, we obtain:

$$S'(q^{fb}(\theta)) = \theta \text{ and } U^{fb}(\theta) = 0 \quad \forall \theta \in \Theta.$$[11]

[10] This is Item 3. in Lemma 1 below.

[11] This outcome is also one possible equilibrium of the intrinsic common agency game when it takes place under complete information. Under complete information, and in sharp contrast with the analysis under asymmetric information that will follow, the principals' non-cooperative behavior need not entail any welfare loss. However, many other inefficient equilibria exist. See Martimort and Stole (2016).

3. IMPLEMENTABILITY, DUALITY AND COOPERATIVE BENCHMARK

Focusing on a model with a continuum of types allows a sharp characterization of the set of implementable allocations by means of familiar incentive and participation constraints.

LEMMA 1 *An allocation (U, q) belongs to $\mathcal{I}$ if and only if:*

1. *$U(\theta)$ is absolutely continuous with at each point of differentiability (i.e., almost everywhere)*

$$\dot{U}(\theta) = -q(\theta) \text{ a.e.,} \tag{3.1}$$

2. *$U(\theta)$ is convex (i.e., $q(\theta)$ is non-increasing),*
3. *$U(\theta)$ induces participation for all types*

$$U(\theta) \geq 0 \quad \forall \theta \in \Theta. \tag{3.2}$$

TARIFFS. Once given an allocation satisfying (3.1) and convexity, simple duality arguments allow us to recover the expression of a nonlinear tariff that implements this allocation. As a first step, we observe that any aggregate contract $T \in \mathcal{T}$ that implements an allocation (U, q) satisfies the following inequality

$$T(q) \leq U(\theta) + \theta q \quad \forall q \in \mathcal{Q}$$

with equality at $q = q(\theta)$. From this, we immediately obtain an upper bound $T^*(q)$ on all implementing contracts as

$$T^*(q) = \min_{\theta \in \Theta} U(\theta) + \theta q \quad \forall q \in \mathcal{Q}. \tag{3.3}$$

In fact, T^* is the least-concave upper semi-continuous tariff implementing (U, q) and thus

$$U(\theta) = \max_{q \in \mathcal{Q}} T^*(q) - \theta q \quad \forall \theta \in \Theta. \tag{3.4}$$

Using the language of convex analysis, the dual conditions (3.3) and (3.4) show that U and T^* are *conjugates* functions. Because T^* is a minimum of linear functions, it is itself concave. Because U is convex, q is non-increasing and we may define the inverse correspondence:

$$\vartheta(q) = \left[\min\{\theta | q = q(\theta)\}, \max\{\theta | q = q(\theta)\}\right].$$

This correspondence is monotone, almost everywhere single-valued and equal to the *subdifferential* of the concave function T^* at any $q \in q(\Theta)$:

$$\partial T^*(q) = \vartheta(q).^{12} \tag{3.5}$$

From Aubin (1998, Theorem 4.3), this *subdifferential* exists on a dense subset of $\mathcal{Q}$ and is almost everywhere single-valued since T^* is concave, implying differentiability there.

[12] While *subdifferentials* are defined for convex functions, the terminology *subdifferential* that would be better suited for concave functions is not so familiar. We thus slightly abuse terminology here and define the *subdifferential* $\partial T(q)$ of a concave function T at q as the following correspondence:

$$\partial T(q) = \{s \in \mathbb{R} | \quad T(x) \leq T(q) + s(x - q) \quad \forall x \in \mathcal{Q}\}$$

Since the high-cost type's participation constraint is binding, a property that holds both in the common agency equilibria explored below and when principals cooperate, we have $U(\theta_2) = 0$. Hence, $T^*(0) = 0$ and the agent is always weakly indifferent between accepting such offer T^* while producing zero output, and refusing to participate.

For further reference, observe also that (3.3) can be written in a more compact form that highlights the following *biconjugacy property*:

$$T^*(q) = \min_{\theta\in\Theta}\left\{\max_{q'\in\mathcal{Q}}\{T^*(q') - \theta q'\} + \theta q\right\}.$$

Broadening the applicability of this requirement, we suggest the following definition.

Definition 3 *An aggregate contract T is* ***biconjugate*** *if and only if $T(0) = 0$ and*

$$T(q) = \min_{\theta\in\Theta}\left\{\max_{q'\in\mathcal{Q}}\{T(q') - \theta q'\} + \theta q\right\} \quad \forall q \in \mathcal{Q}.$$

The set of biconjugate tariffs is denoted by $\mathcal{T}^*$.

An allocation $(\overline{U}, \overline{q}) \in \mathcal{I}^{eq}$ is a biconjugate equilibrium if and only if it is equilibrium implemented by $\overline{T} \in \mathcal{T}^*$. The set of biconjugate equilibria is denoted by $\mathcal{I}^*$.

Remark. Observe that T^* takes finite values since $\mathcal{Q}$ is bounded. T^* is also concave over the convex hull of $q(\Theta)\bigcup\{0\}$ and is linear for intervals of q that lie outside $q(\Theta)$. Consider now the following function T_0, taking values over the extended real line:

$$T_0(q) = \begin{cases} T^*(q) & \text{if } q \in q(\Theta)\bigcup\{0\}\ , \\ -\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

T_0 also implements the allocation (U, q). T_0 differs from T^* in the sense that, had the agent trembled in choosing outputs, choices outside of the equilibrium range $q(\Theta)$ would be severely punished. When $q(\Theta)$ takes only a finite number of values, T_0 is a familiar *forcing contract*. Finally, T_0 inherits of T^*'s concavity over all connected subsets of $q(\Theta)$.

Cooperative outcome. Suppose that principals cooperate in designing contracts. Under asymmetric information, the optimal cooperative allocation $(\overline{U}^c, \overline{q}^c)$ is a solution to:

$$(\mathcal{P}^c): \quad \max_{(U,q)\in\mathcal{I}} E_\theta\left[S(q(\theta)) - \theta q(\theta) - U(\theta)\right].$$

The solution to this standard monopolistic screening problem is now well known. Suppose that types are continuously distributed so that the set of implementable allocations $\mathcal{I}$ is characterized by Lemma 1. Following most of the mechanism design literature, we also impose the following *regularity condition*:[13]

[13]See Bagnoli and Bergstrom (2005).

Assumption 1 *Monotone hazard rate property:*

$$\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \geq 0 \quad \forall \theta \in \Theta.$$

Equipped with this condition, we can remind the well known characterization of the cooperative solution. The cooperative output satisfies:

$$\begin{cases} S'^c(\theta)) = \theta + \frac{F(\theta)}{f(\theta)} & \text{if } S'(0) \geq \theta + \frac{F(\theta)}{f(\theta)}, \\ q^c(\theta) = 0 & \text{otherwise.} \end{cases}$$

The cooperative rent profile is expressed as:

$$U^c(\theta) = \int_{\theta}^{\theta_2} q^c(\tilde{\theta}) d\tilde{\theta}.$$

As requested by Lemma 1, q^c is everywhere non-increasing thanks to Assumption 1 and U^c is convex. The expression of the least-concave (resp. most-concave) nonlinear tariff T^c (resp. T_0^c) that implements this cooperative allocation could be easily recovered by means of (3.3) and (3.5) (resp. (3.6)).

4. EQUILIBRIA AS SOLUTIONS TO SELF-GENERATING PROBLEMS

Martimort and Stole (2012) demonstrate that intrinsic common agency games are aggregate games whose equilibria can be identified with the solution set to *self-generating* optimization problems. Specializing the necessary and sufficient conditions in their Theorem 2' to our present setting, we obtain the following characterization of the *entire* set of equilibrium allocations as solutions of such problems.

Proposition 1 *$(\overline{U}, \overline{q})$ is an equilibrium allocation if and only if it there exists an aggregate tariff $\overline{T}$ satisfying $\overline{T}(0) = 0$ which implements $(\overline{U}, \overline{q})$ and which is such that $(\overline{U}, \overline{q})$ solves the following self-generating maximization problem:*

$$(\overline{\mathcal{P}}) : \quad \max_{(U,q)\in\mathcal{I}} E_\theta \left[S(q(\theta)) - \theta q(\theta) - nU(\theta) + (n-1)(\overline{T}(q(\theta)) - \theta q(\theta)) \right].$$

The maximization problem $(\overline{\mathcal{P}})$ bears some strong similarity with the cooperative mechanism design problem $(\mathcal{P}^c)$. The difference comes from the fact that $(\overline{\mathcal{P}})$ is now *self-generating*: its solution is implemented by an aggregate $\overline{T}$ that also appears in the maximand, embedding the fixed-point nature of equilibrium. Importantly, the fact that intrinsic common agency games are aggregate games allows a significant reduction of the difficulties faced when characterizing such fixed points. Instead of having n optimality conditions determining individual best responses for each principal, only one optimization problem remains after aggregation. This simplification allows us to import the tractable techniques of optimality and buys us sharp results.

In contrast with $(\mathcal{P}^c)$, the objective function in $(\overline{\mathcal{P}})$ now features n-times the extraction of the agent's surplus since each principal, independently of what others are doing, now wants to harvest the agent's information rent. Everything thus happens as if a *surrogate principal* was now in charge of maximizing the principals' collective payoff with the proviso that the agent's information rent is now counted negatively n times. The extra term in the maximand, which represents $n-1$ times the agent's payoff at the induced allocation, captures the fact that a given principal does not take into account the impact of his own contract offer on other principals' payoffs.

Necessity. The necessity part of Proposition 1 can be obtained by summing the individual optimization problems of all principals. An equilibrium allocation, since it maximizes each principal's problem, also maximizes their sum. This summation introduces the n-rent distortion. In any equilibrium with non-zero output, the agent's information rent will thus be overweighted by a factor of n (instead of a coefficient of 1 that would arise had principals cooperated). There is a *"tragedy of the commons"* as the n principals effectively over harvest the agent's information rent, leading to an n-fold marginalization. It is this noncooperative information-rent externality that the principals would like to mitigate in their equilibrium selection; an issue on which we come back in Section 6 below.

Sufficiency. Establishing the sufficiency argument in Proposition 1 is more subtle. Sufficiency bears on the fact that, under intrinsic agency, the objectives of each principal are in fact aligned with those of the surrogate principal in charge of maximizing $(\overline{\mathcal{P}})$. In other words, nothing is lost by aggregating individual objectives. From a more technical point of view, sufficiency is obtained by reconstructing each principal's individual maximization problem from $(\overline{\mathcal{P}})$ itself, so as to align his objectives with those of the surrogate principal. Doing so requires to propose expressions of individual equilibrium tariffs that are derived from the aggregate that solves the self-generating problem $(\overline{\mathcal{P}})$ and that ensures that they are individual best responses to what other principals offer. To this end, consider thus the following individual tariffs:

$$\overline{T}_j(q) = S_j(q) - \frac{1}{n}(S(q) - \overline{T}(q)) \quad \forall j \in \{1, ..., n\}. \tag{4.1}$$

Summing over j yields an aggregate worth $\overline{T}$. Summing instead over all principals but i gives:

$$\overline{T}_{-i}(q) = S_{-i}(q) - \frac{n-1}{n}(S(q) - \overline{T}(q)).$$

By undoing the aggregate offer $\overline{T}_{-i}$ of his competitors so constructed, principal i can always offer any aggregate T he likes, thereby inducing any implementable allocation (U, q). This construction gives principal i an expected payoff of

$$\begin{aligned} &\mathrm{E}_\theta\left[S_i(q(\theta)) - T(q(\theta)) + \overline{T}_{-i}(q(\theta))\right] \\ &\qquad \equiv \mathrm{E}_\theta\left[S(q(\theta)) - T(q(\theta)) - \frac{n-1}{n}(S(q(\theta)) - \overline{T}(q(\theta)))\right], \end{aligned}$$

where the right-hand side equality follows from our previous equation for $\overline{T}_{-i}$. Expressing payments in terms of the agent's rent, we may simplify this payoff as

$$\frac{1}{n}\mathrm{E}_\theta\left[S(q(\theta)) - \theta q(\theta) - nU(\theta) + (n-1)(\overline{T}(q(\theta)) - \theta q(\theta))\right]. \tag{4.2}$$

Up to a positive scalar, the objective function (4.2) exactly replicates that of the surrogate principal. Therefore, principal i's incentives to induce a particular implementable allocation (U, q) are identical to those of the surrogate principal.

REMARK. All principals get the payoff with the above construction:

$$S_i(q) - \overline{T}_i(q) = \frac{1}{n}(S(q) - \overline{T}(q)) \quad \forall i \in \{1, ..., n\}.$$

5. CHARACTERIZATION OF THE EQUILIBRIUM SET WITH CONTINUOUS TYPES

This section fully characterizes the equilibrium set when types are continuously distributed. Importantly, there exist a whole family of equilibrium output allocations beyond the smooth differentiable one that has been the focus of the existing literature.

The solutions to the self-generating problem $(\overline{\mathcal{P}})$ certainly depend on the assumptions made on the distribution function F. Equipped with Assumption 1, we can nevertheless get a sharp characterization of all equilibrium allocations under regularity.

THEOREM 1 *Suppose that types are continuously distributed on $\Theta = [\theta_1, \theta_2]$ according to the cumulative distribution F and that Assumption 1 holds. An allocation $(\overline{U}, \overline{q})$ belongs to $\mathcal{I}^{eq}$ if and only:*

$$\text{(5.1)} \qquad \overline{q}(\theta) \in \arg\max_{q \in \overline{q}(\Theta)} \; S(q) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right) q \quad \forall \theta \in \Theta,$$

$$\text{(5.2)} \qquad \overline{U}(\theta) = \int_{\theta}^{\theta_2} \overline{q}(\tilde{\theta}) d\tilde{\theta} \quad \forall \theta \in \Theta.$$

SURROGATE PRINCIPAL'S INCENTIVE CONSTRAINTS. Condition (5.1) describes the incentive constraints that summarize the surrogate principal's behavior. At any type realization θ, this surrogate principal, whose decisions represent the non-cooperative behavior of the principals, should prefer to induce the choice of the equilibrium output $\overline{q}(\theta)$ rather than any other output that would have been chosen had another type realized.

To evaluate those best choices, the surrogate principal *a priori* considers the maximand of the self-generating problem $(\overline{\mathcal{P}})$. The first remarkable finding is that the surrogate principal's incentive constraints (5.1) are now written *ex post* instead of *ex ante* as in the maximand $(\overline{\mathcal{P}})$. This transformation requires to replace the cost parameter θ by a new expression that entails a n-fold information distortion due to the principals' non-cooperative behavior, namely $\theta + n\frac{F(\theta)}{f(\theta)}$. In the case of a regular problem, Assumption 1 guarantees that this *modified virtual cost parameter* remains non-decreasing and thus $\overline{q}$ is itself non-increasing. Henceforth, any solution to a self-generating problem $(\overline{\mathcal{P}})$ is obtained as the solution to the relaxed problem $(\overline{\mathcal{P}}^r)$ where the convexity requirement for $\overline{U}$ can be omitted.[14]

[14]This feature of the optimization might be lost when the problem is not regular and Assumption 1 fails. We will come back on this issue in Section 7.4 below.

The second remarkable simplification incorporated into (5.1) is that the extra term $(n-1)(\overline{T}(q)-\theta q)$ that is found into the maximand of $(\overline{\mathcal{P}})$ has now disappeared. Intuitively, $\overline{q}(\theta)$ is also a maximizer for this last term since it has to be the agent's equilibrium choice. Although no assumption on differentiability of the equilibrium aggregate tariff $\overline{T}(q)$ is ever made, everything happens as if an envelope condition could be used to simplify the writing of the surrogate principal's incentive constraints.

Equilibrium allocations. The characterization of equilibrium allocations by means of the surrogate principal's incentive constraints (5.1) bears strong similarities with the characterization of implementable allocations found in the literature on mechanism design for delegation problems. (Holmström (1984), Melumad and Shibano (1991), Martimort and Semenov (2006), Alonso and Matouschek (2008), Amador and Bagwell (2013).) This literature demonstrates how a principal can delegate decision-making to a privately informed party in circumstances of conflicting preferences, asymmetric information and when no incentive payments are available. In our context, the conflict of interest comes from the fact that, although principals would like to cooperate, they are unable to do so when each of them can deviate to a bilateral agreement with the agent. The non-cooperative outcome is captured by the optimizing behavior of a surrogate principal. Yet, while cooperating principals maximize a virtual surplus worth

$$(5.3) \qquad S(q) - \left(\theta + \frac{F(\theta)}{f(\theta)}\right) q$$

the surrogate principal cares about a surrogate surplus that entails the *modified virtual cost parameter*

$$(5.4) \qquad S(q) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right) q.$$

This surrogate surplus accounts for the fact that non-cooperating principals extract n times the agent's rent while cooperating principals only care about extracting that rent once; a source of conflict.

Although there is no asymmetric information *per se* between the cooperating principals and their surrogate, the latter implements at equilibrium an allocation which is a *pointwise* optimum of the surrogate surplus. This maximization thus induces a set of incentive constraints that are reminiscent of those found in the aforementioned literature. Everything happens thus as if the surrogate principals was informed on the agent's cost himself although he replaces this cost parameter by its non-cooperative virtual version.

Borrowing techniques from the delegation literature gives us a clear characterization of equilibrium outputs. As a preamble and for further reference, we shall denote by q^m the surrogate principal's ideal output, i.e., the unconstrained maximum of the strictly concave objective (5.4).

Maximal Equilibrium. Following a path taken by Martimort and Stole (2015) in their analysis of delegated common agency games, one may refine among all equilibria described in Theorem 1 by imposing the requirement that the maximization in (5.1) is taken over the full domain $\mathcal{Q}$:

$$q^m(\theta) \in \arg\max_{q\in\mathcal{Q}} \; S(q) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right) q \quad \forall\theta \in \Theta.$$

The corresponding condition characterizes the surrogate's ideal output which, thanks to Assumption 1, is non-increasing:

$$(5.5) \qquad \begin{cases} S'^m(\theta)) = \theta + n\frac{F(\theta)}{f(\theta)} & \text{if } S'(0) \geq \theta + n\frac{F(\theta)}{f(\theta)}, \\ q^m(\theta) = 0 & \text{otherwise.} \end{cases}$$

Define now the rent allocation U^m as:

$$U^m(\theta) = \int_\theta^{\theta_2} q^m(\tilde{\theta})d\tilde{\theta}.$$

We can now prove the following important result.

Proposition 2 *Suppose that types are continuously distributed on $\Theta = [\theta_1, \theta_2]$ according to the cumulative distribution F and that Assumption 1 holds. The maximal allocation (U^m, q^m) is an equilibrium.*

Observe that any equilibrium allocation that satisfies the less demanding requirement (5.1) is equal to the maximal equilibrium on any interval where it is continuous and separating. The qualifier *maximal* can thus be understood as imposing that smoothness requirement over the whole type space.

Compared with a cooperative outcome, the maximal equilibrium allocation features a distortion which is now proportional to n times the hazard rate. This captures the fact that, at equilibrium, each principal adds his own distortion for rent extraction reasons.

This maximal equilibrium has been the focus of the earlier common agency literature. For instance, Martimort and Stole (2012) used that particular selection to prove existence of an equilibrium to intrinsic common agency games under broad conditions although no full characterization of the whole equilibrium set was given, in sharp contrast with the present paper. In more applied works, Laffont and Tirole (1993, Chapter 17) modeled privatization as a common agency game between shareholders and regulators controlling the firm's manager. Their conclusion that joint control leads to low-powered incentives relies on the selection of the smooth maximal equilibrium and is thus fragile.

More detailed characterization. Next proposition provides a complete and more detailed characterization of all equilibrium output profiles.

Proposition 3 *Suppose that types are continuously distributed on $\Theta = [\theta_1, \theta_2]$ according to the cumulative distribution F and that Assumption 1 holds.*

- Necessary conditions.
  1. *Any equilibrium output profile $\overline{q}$ is non-increasing.*

2. *At any point θ of differentiability $\overline{q}$, i.e., almost everywhere, the following condition holds:*

$$\dot{\overline{q}}(\theta)\left(S'(\overline{q}(\theta))-\theta-n\frac{F(\theta)}{f(\theta)}\right)=0. \tag{5.6}$$

3. *At any interior point of discontinuity $\theta_0\in(\theta_1,\theta_2)$, $\overline{q}$ is right- and left-continuous and the surrogate surplus remains continuous:*

$$S(\overline{q}(\theta_0^+))-\left(\theta_0+n\frac{F(\theta_0)}{f(\theta_0)}\right)\overline{q}(\theta_0^+)=S(\overline{q}(\theta_0^-))-\left(\theta_0+n\frac{F(\theta_0)}{f(\theta_0)}\right)\overline{q}(\theta_0^-). \tag{5.7}$$

- Sufficient conditions. *Any non-increasing output profile $\overline{q}$ satisfying (5.6) outside discontinuities and (5.7) at any discontinuity point is an equilibrium allocation.*

Equilibrium Outputs. From (5.6), any equilibrium output profile is either flat over some range, in which case it is unresponsive to the agent's private information or, when it is decreasing in θ, it corresponds to the maximal equilibrium output. To illustrate using an example that will be relevant in what follows, an equilibrium output profile can be obtained simply by putting a floor on the maximal equilibrium output. In that case, principals are unable to implement outputs which are too low.

Discontinuities in the equilibrium output $\overline{q}$ have also a quite specific structure. First, the fact that $\overline{q}$ is non-increasing implies that such discontinuities are countable in number. Second, such discontinuities must preserve the necessary continuity of the surrogate surplus:

$$\max_{q\in q(\Theta)} S(q)-\left(\theta+n\frac{F(\theta)}{f(\theta)}\right)q.$$

This requirement implies that outputs on both sides of such discontinuities satisfy the simple optimality condition (5.7), which expresses the fact that the surrogate principal should be indifferent between moving output on either side of the gap.

Equilibrium Ranges. Since q^m satisfies Condition (5.6) and is non-increasing when Assumption 1 holds, it is an equilibrium outcome. Introducing a discontinuity of $\overline{q}$ at a given type θ_0 amounts to having a gap $(q^m(\theta_0^+),q^m(\theta_0^-))$ in the the range of q^m to obtain the range of $\overline{q}$. This is done by imposing sufficiently negative payments for any output in this gap. Each principal is willing to introduce such payments if other principals are expected to do so. In this manner, we may generate arbitrary equilibrium allocations by introducing gaps in the maximal allocation. This allocation is called *maximal* precisely because, in the sense of set inclusion, it contains the ranges of all discontinuous equilibria.

Another immediate consequence of Proposition 3 is that for any subset $\overline{Q}$ of $q^m(\Theta)$ which is the union of a countable number of connected intervals, there is a unique equilibrium allocation $\overline{q}$ whose range is $\overline{Q}$ itself. A specific case arises when principals offer forcing contracts at a finite number of outputs. Even though they do not provide any in-depth analysis of those equilibria, Laffont and Tirole (1993, Chapter 17) already devoted an appendix to discuss an interesting subclass of equilibria which are implemented by means of forcing contracts. Their analysis is unfortunately incomplete. Forcing contracts induce allocations with exhibit bunching almost everywhere but they only represent a special case of the more complete analysis of Theorem 1 and Proposition 3 above. Moreover, and in sharp contrast with ours, their analysis does not allow welfare comparison of those non-differentiable equilibria with the smooth maximal allocation.

TARIFFS. An aggregate payment $\overline{T}$ are easily reconstructed from any equilibrium allocation $(\overline{U}, \overline{q})$ where $\overline{q}$ satisfies (5.6) outside discontinuities and (5.7) at any discontinuity point. First, using the rent profile $\overline{U}$ obtained from (5.2), the simple duality argument in (3.3) gives us a nonlinear price T^* that implements $(\overline{U}, \overline{q})$. Second, principals are prevented from deviating to a contract that would induce the agent to choose outputs within a discontinuity gap by imposing that the aggregate tariff $\overline{T}$ entails infinitely negative payments for $q \notin \overline{q}(\Theta) \bigcup \{0\}$ as requested from (3.6). Finally, conditions (4.1) allow to reconstruct from this aggregate tariff the equilibrium tariffs offered by each principal.

COMPARISON WITH THE COOPERATIVE OUTCOME. It is interesting to ascertain the validity of our findings in Theorem 1 and Proposition 3 in the limiting case where $n = 1$. Of course, the maximal equilibrium allocation corresponds to the cooperative solution under those circumstances. Although the cooperative solution satisfies the necessary conditions laid out in Theorem 1 and Proposition 3, sufficiency no longer holds. In fact, when $n > 1$, some of the equilibrium allocations are obtained by the very fact that a given principal may not be free to choose any output because other principals have stipulated infinitely negative payments at this output; a threat that is only available when $n > 1$.

A SIMPLE NUMERICAL EXAMPLE. As an illustration of the various allocations that were characterized above, let consider the following quadratic-uniform example. Suppose that $n = 2$, $S_1(q) = S_2(q) = 4q - \frac{1}{4}q^2$, and θ is distributed uniformly on $[1, 5]$ so that $\frac{F(\theta)}{f(\theta)} = \theta - 1$. It is straightforward to derive the allocations for the first-best outcome, the cooperative optimum, and the maximal noncooperative equilibrium as:

$$q^{fb}(\theta) = 8 - \theta, \qquad q^c(\theta) = \max\{9 - 2\theta, 0\}, \qquad q^m(\theta) = \max\{10 - 3\theta, 0\}.$$

The (aggregate) tariff for the cooperative solution and the maximal equilibrium are respectively given by:

$$T^c(q) = \frac{9}{2}q - \frac{q^2}{4} \text{ and } T^m(q) = \frac{10}{3}q - \frac{q^2}{6}.$$

The range of the maximal equilibrium is $q^m(\Theta) = [0, 7]$. Consider now taking off some outputs in that range so as to construct an equilibrium whose range is $\overline{q}(\Theta) = q^m(\Theta)/(2, 5)$. It is straightforward to check that $\overline{q}$ is discontinuous at $\theta_0 = \frac{13}{6}$ which is indifferent between moving on either side of the discontinuity. In this uniform-quadratic setting, we can also verify that an implementing aggregate tariff satisfies:

$$\overline{T}(q) = \begin{cases} T^m(q) & \text{if } q \in q^m(\Theta)/(2,5), \\ -\infty & \text{otherwise.} \end{cases}$$

■

6. EQUILIBRIUM SELECTION

We now present two approaches to possibly select within the plethora of equilibria found above. The first one is based on the attractive property of biconjugacy. The second is a more standard Pareto criterion.

6.1. *Maximal Equilibrium and Biconjugacy*

The maximal equilibrium is obviously equilibrium implementable by a biconjugate tariff $T^m(q) = \min_{\theta\in\Theta} U^m(\theta) + \theta q$. Following (3.6), this equilibrium could also be sustained by a tariff imposing infinitely negative payments for outputs that would not belong to $q^m(\Theta)\bigcup\{0\}$. *A contrario*, other non-maximal equilibria with discontinuity holes are equilibrium implementable with nonlinear tariffs $\overline{T}$ satisfying (3.6) but cannot be equilibrium implemented with the corresponding biconjugate tariff $T^*(q) = \min_{\theta\in\Theta} \overline{U}(\theta) + \theta q$.

To see why, consider the equilibrium allocation obtained by withdrawing $(q^m(\theta_0^+), q^m(\theta_0^-))$ from $q^m(\Theta)$ and suppose that $\overline{T}$ is replaced by T^*. Adding more attractive options for $q \in (q^m(\theta_0^+), q^m(\theta_0^-))$ of course does not change the agent's behavior. Only the agent with type θ_0 is willing to take those options. Given that T^* is the least-concave tariff, it is linear over $(q^m(\theta_0^+), q^m(\theta_0^-))$ and type θ_0 is actually indifferent between all options in $(q^m(\theta_0^+), q^m(\theta_0^-))$. Yet, with T^*, filling the discontinuity gap makes it now attractive for at least one principal to deviate and propose options within the gap. In other words, replacing $\overline{T}$ by T^* changes the solution to the self-generating problem $(\overline{\mathcal{P}})$ and the allocation $(\overline{U}, \overline{q})$ is no longer the solution sustained by T^*. Imposing biconjugacy on the aggregate tariff thus refines the equilibrium set.

PROPOSITION 4 *Suppose that types are continuously distributed on $\Theta = [\theta_1, \theta_2]$ according to the cumulative distribution F and that Assumption 1 holds. The maximal equilibrium (U^m, q^m) is the only biconjugate equilibrium.*

6.2. *Ex Ante Optimal Equilibrium*

Another possibility is to select among equilibria in terms of the expected net surplus they give to the principals. Indeed, if principals could meet *ex ante* and negotiate over the equilibrium to be played, a reasonable prediction would be that they would agree to play the equilibrium that maximizes their *ex ante* collective payoff. Therefore, we now investigate what is this *ex ante* best equilibrium allocation for the principals. In this respect, Proposition 3 shows for any set $\overline{\mathcal{Q}}$ such that $\overline{\mathcal{Q}} \subset q^m(\Theta)$ and $\overline{\mathcal{Q}}$ is a union of countable intervals, there is a unique equilibrium allocation, $\overline{q}$ such that $\overline{q}(\Theta) = \overline{\mathcal{Q}}$. This allocation fully defines the corresponding aggregate transfer as we have just seen. Thus, the equilibrium selection problem can be reduced to the principals optimally choosing a delegation set $\overline{\mathcal{Q}}$ to offer to the surrogate delegate, who then chooses an allocation solving a self-generating maximization program $(\overline{\mathcal{P}})$ where the equilibrium tariff $\overline{T}$ has domain $\overline{\mathcal{Q}}$. Restated in this form, we may apply a recent result from Amador and Bagwell (2013),[15] to conclude that the optimal delegation set is a connected interval putting a floor on outputs. To do so, it is sufficient to make the following assumption on F.

[15]See Martimort and Semenov (2006) and Alonso and Matoushek (2008) for earlier slightly stronger conditions along those lines.

Assumption 2 *For almost all* $\theta \in \Theta$

$$\left(n\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right)+1\right)\left(\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right)+1\right) \geq (n-1)\frac{F(\theta)}{f(\theta)}\frac{d^2}{d\theta^2}\left(\frac{F(\theta)}{f(\theta)}\right).$$

In tandem with Assumption 1, Assumption 2 requires that the inverse hazard rate is not too convex. It is satisfied by several well-known distributions with positive supports, including uniform, exponential, Laplace, Pareto, Weibull and Chi-square.[16]

Proposition 5 *Suppose that* F *satisfies Assumptions 1 and 2. The principals' best equilibrium from an ex ante point of view is characterized by an output interval,* $\overline{\mathcal{Q}} = [q^m(\hat{\theta}), q^m(\theta_1)]$*, with* $\hat{\theta} > \theta_1$ *and*

$$E_\theta\left[S'^m(\hat{\theta})) - \theta - \frac{F(\theta)}{f(\theta)}|\theta \geq \hat{\theta}\right] = 0. \tag{6.1}$$

When $q^m(\theta_2) > 0$*, we have* $\hat{\theta} > \theta_1$ *and the maximal equilibrium is never ex ante optimal.*

Proposition 5 tells us that the range of the maximal equilibrium, $q^m(\Theta)$, generates too much informational-rent distortion from an *ex ante* point of view for the principals. They would prefer to put a floor on the set of equilibrium outputs, thereby reducing the sensitivity of the output allocation on the agent's underlying type, in order to mitigate the problem of over-marginalization. Such a floor can be implemented by an aggregate tariff which sets $\overline{T}(q) = -\infty$ for any $q \in (0, q^m(\hat{\theta}))$.

Numerical example (continued). Using the definition of q^m given in (5.5), Condition (6.1) can be written in terms of the cut-off $\hat{\theta}$ only as:

$$E_\theta\left(2\theta - 1|\theta \geq \hat{\theta}\right) = \min\left\{8, 3\hat{\theta} - 2\right\}.$$

Tedious computations show that the cut-off is $\hat{\theta} = 3$. The output profile at the *ex ante* best non-cooperative equilibrium is a truncation of the maximal equilibrium profile:

$$\overline{q}(\theta) = \max\{1, 10 - 3\theta\}.$$

At that best equilibrium, the principals shut down payments for q less than $\hat{q} = q^m(3) = 1$. Bunching arises over the upper tail of the distribution $[3, 5]$:

$$\overline{T}(q) = \begin{cases} 0 & \text{if } q = 0, \\ \frac{11}{6} + \frac{10}{3}q - \frac{q^2}{6} & \text{if } q \in [1, 7], \\ -\infty & \text{otherwise.} \end{cases}$$

Observe also that $\hat{q} > q^m(5) = 0$ so that, by restricting the equilibrium set of outputs principals are able to implement outputs that sometimes above the cooperative solution.■

7. TWO-TYPES MODELS

The goal of this section is twofold. First, we now show that the techniques that were developed above to characterize equilibria with continuous types can also be readily used with discrete types. This set-up is of course of much interest in view of applied works. Second, we bridge a gap between the continuous and discrete types model and show how the equilibrium selection in the latter can be justified by a careful analysis of the former.

[16]This property is verified for the Weibull and Chi-square distributions using numerical methods.

7.1. *Benchmark: The Cooperative Outcome*

The solution to the monopolistic screening problem is well known in two-type screening models.[17] The low-cost agent produces the first-best output, $q^c(\theta_1) = q^{fb}(\theta_1)$, and gets an information rent worth $U^c(\theta_1) = \Delta\theta q^c(\theta_2)$. In contrast, the high-cost agent gets no rent, $U^c(\theta_2) = 0$ and produces an output $q^c(\theta_2)$, less than the first-best level $q^{fb}(\theta_2)$:

$$(7.1) \qquad \begin{cases} S'(q^c(\theta_2)) = \theta_2 + \frac{\nu}{1-\nu}\Delta\theta & \text{if } S'(0) \geq \theta_2 + \frac{\nu}{1-\nu}\Delta\theta, \\ q^c(\theta_2) = 0 & \text{otherwise.} \end{cases}$$

The cooperative outcome (U^c, q^c) can again be implemented with a variety of aggregate tariffs. A first possibility is the *biconjugate* contract T^*:

$$T^*(q) = \min_{\theta \in \Theta} U^c(\theta) + \theta q = \min\left\{U^c(\theta_1) + \theta_1 q; \theta_2 q\right\}.$$

Another one is the following *forcing contract*:

$$\overline{T}^c(q) = \begin{cases} \theta_2 q^c(\theta_2) & \text{if } q = q^c(\theta_2), \\ \theta_1 q^c(\theta_1) + \Delta\theta q^c(\theta_2) & \text{if } q = q^c(\theta_1), \\ 0 & \text{if } q = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

7.2. *Characterization of the Set of Equilibria*

Our next Theorem describes the set of *equilibrium implementable* allocations. It provides a discrete counterpart to Theorem 1.

THEOREM 2 *Suppose that types are discretely distributed on $\Theta = \{\theta_1, \theta_2\}$. An allocation $(\overline{U}, \overline{q})$ belongs to $\mathcal{I}^{eq}$ if and only if:*

$$(7.2) \qquad S(\overline{q}(\theta_1)) - \theta_1\overline{q}(\theta_1) \geq S(\overline{q}(\theta_2)) - \theta_1\overline{q}(\theta_2);$$

$$(7.3) \qquad S(\overline{q}(\theta_2)) - \left(\theta_2 + \frac{n\nu}{1-\nu}\Delta\theta\right)\overline{q}(\theta_2)$$
$$\geq \max\left\{S(\overline{q}(\theta_1)) - \left(\theta_2 + \frac{n\nu}{1-\nu}\Delta\theta\right)\overline{q}(\theta_1) - (n-1)\Delta\theta(\overline{q}(\theta_1) - \overline{q}(\theta_2)); 0\right\};$$

$$(7.4) \qquad \overline{U}(\theta_1) = \Delta\theta\overline{q}(\theta_2) \geq \overline{U}(\theta_2) = 0.$$

SURROGATE PRINCIPAL'S INCENTIVE COMPATIBILITY. Condition (7.4) is familiar. At the optimum of $(\overline{\mathcal{P}})$, the low-cost agent's incentive constraint and the high-cost agent's participation constraints are necessarily binding. When turning to the optimal conditions with respect to outputs, we recognize that (7.2) and (7.3) represent the surrogate principal's incentive constraints. These constraints bear much similarity with the continuum case; the difference being that deviations are now non-local and differentiability is lost

[17]See Laffont and Martimort (2002, Chapter 2) for instance.

in expressing those incentive constraints. These constraints are obtained by imposing the minimal *proviso* that the surrogate principal should prefer to induce the equilibrium output profile $(\overline{q}(\theta_1), \overline{q}(\theta_2))$ rather than inducing the agent to either switch for equilibrium actions corresponding to different types or to non-participation.

Monotonicity. Simple revealed preferences arguments show that any output profile that satisfy conditions (7.2) and (7.3) is necessarily non-increasing:

$$\overline{q}(\theta_1) \geq \overline{q}(\theta_2). \tag{7.5}$$

Monotonicity appears as an implicit requirement of the constrained set $\mathcal{I}$ for $(\overline{\mathcal{P}})$. Yet, even if the surrogate principal were to maximize a relaxed problem $(\overline{\mathcal{P}}^r)$ which would not be constrained by such monotonicity condition, the requirement of self-generation would impose the surrogate principal's incentive constraints (7.2) and (7.3). These conditions, in turn, imply monotonicity and thus the solution of the relaxed problem also solves the more constrained problem $(\overline{\mathcal{P}}^r)$.

Implementation with forcing contracts. An aggregate forcing contract suffices to implement any equilibrium allocation satisfying conditions in Theorem 2:

$$\overline{T}(q) = \begin{cases} \theta_2\overline{q}(\theta_2) & \text{if } q = \overline{q}(\theta_2), \\ \theta_1\overline{q}(\theta_1) + \Delta\theta\overline{q}(\theta_2) & \text{if } q = \overline{q}(\theta_1), \\ 0 & \text{if } q = 0, \\ -\infty & \text{otherwise.} \end{cases} \tag{7.6}$$

Equilibrium offers, $\overline{T}_i$, are then reconstructed using (4.1). In particular, all those individual offers become also forcing contracts with harsh punishments off path.

Numerical Example (continued). The cooperative allocation (U^c, q^c) can sometimes be supported as an equilibrium with forcing contracts. To illustrate, consider our numerical example, simply modified by the fact that types are now distributed on on $\{1, 5\}$ with probabilities $\nu = 0.2$ and $1 - \nu = 0.8$. Then, $q^c(1) = 7$ and $q^c(5) = 2$. The cooperative allocation (U^c, q^c) belongs to $\mathcal{I}^{eq}$ if (7.3) holds which amounts to

$$3q^c(5) - \frac{(q^c(5))^2}{2} - nq^c(5) \geq \max\left\{0; 3q^c(1) - \frac{(q^c(1))^2}{2} - nq^c(1) - 20(n-1)\right\}.$$

This condition holds when $n = 2$; a case where we could have expected distortions to already arise from the principals' non-cooperative behavior. This suggests that the set $\mathcal{I}^{eq}$ may be too large to convey the basic intuition that the principals' non-cooperative behavior may entail some welfare cost for them. Section 7.3 provides conclusions which are more in lines with what intuition commands by restricting the space of contracts. ■

Ex ante Optimal Equilibrium. To focus on the interesting case with welfare costs, we shall assume that the cooperative outcome cannot be supported at equilibrium.

Assumption 3

$$S(q^c(\theta_2)) < \left(\theta_2 + \frac{n\nu}{1-\nu}\Delta\theta\right) q^c(\theta_2).$$

Assumption 3 holds when n is sufficiently large and non-cooperative behavior becomes very costly. Next Proposition shows that the best equilibrium is nevertheless chosen to be as close as possible to this cooperative solution.

PROPOSITION 6 *Suppose that types are discretely distributed on $\Theta = \{\theta_1, \theta_2\}$ and Assumption 3 holds. The best allocation $(\hat{U}, \hat{q})$ in $\mathcal{I}^{eq}$ from the principals' ex ante point of view has the low-cost agent producing at the first-best $q^{fb}(\theta_1)$ and the high-cost agent's output being less than the cooperative outcome; $\hat{q}(\theta_2) < q^c(\theta_2)$, where*

$$(7.7) \qquad S(\hat{q}(\theta_2)) = \left(\theta_2 + \frac{n\nu}{1-\nu}\Delta\theta\right)\hat{q}(\theta_2).$$

When n increases, $\hat{q}(\theta_2)$ decreases. In other words, principals face greater difficulties to coordinate and contractual externalities are exacerbated as their number increases.

7.3. *Biconjugate Equilibria*

MOTIVATION. The forcing contracts described in (7.6) help principals to coordinate. We now show how deviations can instead be facilitated and the set of equilibrium allocations can be refined when contracts entail less severe punishments off path. This is particularly the case when the aggregate contract satisfies biconjugacy.

CHARACTERIZATION. Imposing biconjugacy provides a sharp characterization of equilibrium allocations. In a two-type context, the aggregate transfer must satisfy:

$$(7.8) \qquad \overline{T}(q) = \min_{\theta\in\Theta} \overline{U}(\theta) + \theta q = \min\left\{\overline{U}(\theta_1) + \theta_1 q; \theta_2 q\right\}.$$

Biconjugate tariffs admit subdifferentials at equilibrium points, respectively given by $\partial\overline{T}(\overline{q}(\theta_2)) = [\theta_1, \theta_2]$ and $\partial\overline{T}(\overline{q}(\theta_1)) = \{\theta_1\}$. This allows us to use convex calculus to characterize equilibrium allocations as solutions to a self-generating problem $(\overline{\mathcal{P}})$. As a consequence, the *non-local* optimality conditions (7.2) and (7.3) will now be replaced by more restrictive *marginal* optimality conditions (resp. (7.9) and (7.10) below). These conditions are similar to those obtained for a continuum of types.

THEOREM 3 *Suppose that types are discretely distributed on $\Theta = \{\theta_1, \theta_2\}$. An allocation $(\overline{U}, \overline{q})$ belongs to $\mathcal{I}^*$, if and only if (7.4) holds and*

$$(7.9) \qquad S'(\overline{q}(\theta_1)) = \theta_1,$$

$$(7.10) \qquad 0 \in S'(\overline{q}(\theta_2)) - \left(\theta_2 + \frac{n\nu}{1-\nu}\Delta\theta\right) + (n-1)([\theta_1, \theta_2] - \theta_2).$$

By restricting attention to biconjugate equilibria, inefficiencies for the low-cost type are now ruled out whereas such inefficiencies are possible in equilibria which are not biconjugate as demonstrated by Theorem 2. As a result, $\mathcal{I}^*$ is a strict subset of $\mathcal{I}^{eq}$.

Biconjugate equilibria also feature distortions below the cooperative outcome for the high-cost agent. All biconjugate equilibrium outputs $\overline{q}(\theta_2)$ must lie within an interval $[\tilde{q}(\theta_2), q^*(\theta_2)]$, whose upper $q^*(\theta_2)$ and lower bound $\tilde{q}(\theta_2)$ are respectively defined by

$$(7.11)\quad S'(q^*(\theta_2)) = \theta_2 + \frac{n\nu}{1-\nu}\Delta\theta \text{ and } S'(\tilde{q}(\theta_2)) = \theta_2 + \left(\frac{n\nu}{1-\nu} + n - 1\right)\Delta\theta.$$ [18]

The existence of a continuum of biconjugate equilibria stands in contrast with the continuum case where unicity of such equilibria prevailed. Multiplicity comes from the fact that, at the equilibrium output $\overline{q}(\theta_2)$, a biconjugate tariff admits a subdifferential, $\partial\overline{T}(\overline{q}(\theta_2)) = [\theta_1, \theta_2]$ which is not single-valued. Inserting this this subdifferential into the optimality condition for the self-generating problem leaves a whole interval of outputs as possible equilibria. The upper bound of this set, that now corresponds to the lesser output distortion $q^*(\theta_2)$, is the best biconjugate equilibrium from the principals' viewpoint.

Proposition 7 *The best allocation (U^*, q^*) in $\mathcal{I}^*$ from the principals' ex ante point of view has the low-cost agent producing at the first-best $q^{fb}(\theta_1)$ and the high-cost agent's output being more distorted than the cooperative outcome; $q^*(\theta_2) < q^c(\theta_2)$.*

The definitions of the best equilibrium outputs in the full set $\mathcal{I}^{eq}$, namely (7.7), and in the *biconjugate* domain $\mathcal{I}^*$, (7.11), bear some strong similarities. Moving from $\mathcal{I}^{eq}$ to $\mathcal{I}^*$, the *non-local* definition (7.7) of the best equilibrium is again replaced by a marginal condition (7.11). Qualitative properties of the two solutions are also closely related to the extent that the marginal aggregate surplus $S'(q)$ and the average aggregate surplus $S(q)/q$ vary in similar direction. Regardless of whether modelers choose to select the best equilibrium within $\mathcal{I}^{eq}$ or $\mathcal{I}^*$, comparative statics share a likeness.

The restriction to biconjugate equilibria prevents the principals from reaching the highest possible equilibrium collective payoff that could be reached with a forcing aggregate of the kind (7.6). This fact is probably best seen when Assumption 3 holds. Then, the *ex ante* optimal equilibrium allocation in $\mathcal{I}^{eq}$ is obtained at $(\hat{U}, \hat{q})$. Yet, the strict concavity of $S(q)$, together with the fact that $\hat{q}(\theta_2)$ is positive, implies:

$$S'(\hat{q}(\theta_2)) \leq \frac{S(\hat{q}(\theta_2))}{\hat{q}(\theta_2)} = \theta_2 + \frac{n\nu}{1-\nu}\Delta\theta = S'(q^*(\theta_2)) \Rightarrow \hat{q}(\theta_2) > q^*(\theta_2).$$

In other words, the best equilibrium is never a biconjugate equilibrium. Yet, the best biconjugate equilibrium exhibits the same properties as the maximal equilibrium of the continuum case. Under both scenarios, outputs entail a n-fold informational distortion.

7.4. *From Continuous to Discrete Models: A Justification for Biconjugacy*

Our goal in this section is to rationalize the focus on *biconjugate* equilibria of the discrete model. We show that those equilibria are actually limits of equilibria obtained in the continuous types model as a conveniently chosen family of continuous distributions converges towards the point mass distribution of the discrete model. Unfortunately, along such sequence, distribution functions no longer satisfy the *Monotone Hazard Rate Property*. Therefore, solutions to self-generating problems are no longer regular. Bunching is

[18] For future reference, we may define the allocation (U^*, q^*) with $U^*(\theta_1) = \Delta\theta q^*(\theta_2) > U^*(\theta_2) = 0$ by requesting the low-cost agent to produce the first-best output level $q^*(\theta_1) = q^{fb}(\theta_1)$.

pervasive along such sequences and the maximal equilibrium as expressed in (5.5) no longer exists. To analyze limits equilibria, we are thus obliged to derive solutions to self-generating problems without assuming regularity. It turns out that the limits of those equilibria with bunching are *biconjugate* equilibria of the discrete model. We believe that this limit argument gives thus some foundations for looking at *biconjugate* equilibria in discrete models, and maybe more specifically to recommend the choice of the best equilibrium (U^*, q^*) within that class as a practical tool for modelers.[19]

As far as convergence is concerned, there exists a plethora of distribution functions that fail to satisfy the *Monotone Hazard Rate Property* but still converge towards the distributions of the discrete models on Θ putting charges ν at θ_1 and $1-\nu$ at θ_2 respectively. We now list attractive properties that the sequences under scrutiny should feature.

Assumption 4 *F_k is continuous, strictly increasing over Θ with $F_k(\theta_1) = 0$, $F_k(\theta_2) = 1$ for all $k \in \mathbb{N}$,*

$$\lim_{k\to+\infty} F_k(\theta) = \nu \text{ and } \lim_{k\to+\infty} \frac{F_k(\theta)}{f_k(\theta)} = +\infty \quad \forall \theta \in (\theta_1, \theta_2),$$

and the virtual cost $z_k(\theta) = \theta + \frac{F_k(\theta)}{f_k(\theta)}$ is hill-shaped for k large enough.

Assumption 4 is satisfied by some sequences that all show that the continuous types model can thus be viewed as a convenient approximation of the discrete types model. To illustrate with one example, let F_k $(k \in \mathbb{N})$ be defined as:

$$F_k(\theta) = \frac{2\nu}{1 + \frac{1}{k} log\left(\frac{\theta_2-\theta}{\theta-\theta_1}\right) + \sqrt{\left(1 + \frac{1}{k} log\left(\frac{\theta_2-\theta}{\theta-\theta_1}\right)\right)^2 - \frac{4\nu}{k} log\left(\frac{\theta_2-\theta}{\theta-\theta_1}\right)}} \quad \forall \theta \in \Theta. \tag{7.12}$$

It is straightforward to check that the sequence F_k satisfies the first two conditions in Assumption 4. Turning to the third one, observe that the distributions F_k do not satisfy Assumption 1 when k is large enough. Indeed, tedious computations show that:

$$\frac{F_k(\theta)}{f_k(\theta)} \approx_{k\to+\infty} \frac{k}{(1-\nu)\Delta\theta}(\theta_2 - \theta)(\theta - \theta_1) \quad \forall \theta \in (\theta_1, \bar{\theta}). \tag{7.13}$$

In the limit, the hazard rate is thus hill-shaped (being thus non-monotonic) and converges towards infinity. This implies that the maximal allocation (5.5) corresponding to the distribution F_k is increasing over some range and thus can longer be an equilibrium. This non-monotonicity of the hazard rate implies that bunching is now innocuous.

To characterize the whole set of equilibria that may arise with non-monotonic hazard rates, we have thus to follow the *ironing procedure* proposed by Myerson (1981) in his seminal study of optimal auctions when virtual valuations are non-monotonic.

Theorem 4 characterizes the limits of sequences of equilibria of the continuous model corresponding to distributions satisfying Assumption 4. Bunching arises along the sequences. Those limits necessarily lies in the set of *biconjugate* equilibria of the discrete two-types model. This gives us some motivation for looking those equilibria in discrete environments.

[19] Yet, we must also acknowledge that our results below do not claim that equilibria sets corresponding to all converging sequences of distributions of the continuous model would converge towards the biconjugate equilibria of the discrete type model.

THEOREM 4 *Suppose that types are continuously distributed on Θ according to cumulative distributions F_k satisfying Assumption 4. Any converging sequence of equilibrium allocations $\overline{q}_k$ which is separating on a right-neighborhood of θ_1 has for limit an equilibrium allocation of the two-type discrete common agency game which is implemented with biconjugate tariffs:*

$$\lim_{k \to +\infty} \overline{q}_k(\theta) = \begin{cases} q^{fb}(\theta_1) & \text{if } \theta = \theta_1 \ , \\ \overline{q}^p_\infty \in [\tilde{q}(\theta_2), q^*(\theta_2))] & \text{otherwise.} \end{cases} \tag{7.14}$$

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Proof of Lemma 1: The proof is standard and thus omitted. See Rochet (1987) or Milgrom and Segal (2002). *Q.E.D.*

Proof of Proposition 1: The proof follows similar steps to those in Martimort and Stole (2012, Theorem 2'), though we explicitly treat the agent's participation decision, δ, here for completeness.

Necessity. Given the aggregate tariff $\overline{T}_{-i}$ offered by competing principals, principal i's net gain with the agent of type θ when $(q, \delta) \in \mathcal{Q} \times \{0, 1\}$ is chosen is given by

$$S_i(q) - \delta\overline{T}(q) \equiv S_i(q) - \theta q + \delta\overline{T}_{-i}(q) - (\delta\overline{T}(q) - \theta q).$$

For $(\overline{U}, \overline{q}, \overline{\delta})$ to be an equilibrium allocation, it must be that principal i desires to implement this allocation which must thus solve:

$$(\overline{U}, \overline{q}, \overline{\delta}) \in \arg\max_{(U,q,\delta)\in\mathcal{I}} E_\theta[S_i(q(\theta)) - \theta q(\theta) + \delta(\theta)\overline{T}_{-i}(q(\theta)) - U(\theta)].$$

Note that every principal i faces the same domain of maximization, $\mathcal{I}$; the difference between the programs of any two principals, i and j, is entirely embedded in the differences in the aggregates $\overline{T}_{-i}$ and $\overline{T}_{-j}$. Following Martimort and Stole (2012), an equilibrium allocation must necessarily maximize the sum of the principals' programs. Thus, $(\overline{U}, \overline{q}, \overline{\delta})$ must solve:

$$\text{(A1)} \quad (\overline{U}, \overline{q}, \overline{\delta}) \in \arg\max_{(U,q,\delta)\in\mathcal{I}} E_\theta\left[S(q(\theta)) - \theta q(\theta) - nU(\theta) + (n-1)(\delta(\theta)\overline{T}(q(\theta)) - \theta q(\theta))\right].$$

Note that the solution to this problem has always at least one type, say $\hat{\theta}$, such that

$$\text{(A2)} \quad \overline{U}(\hat{\theta}) = 0.$$

Indeed, if it was not the case, then the whole rent profile could be reduced uniformly by $\epsilon > 0$ without modifying output and this modification would improve the value of the program.

From a remark in the text, any aggregate tariff that implements an equilibrium $(\overline{U}, \overline{q})$ solution to the self-generating problem above must satisfy:

$$T(q) \leq T^*(q) = \min_{\theta\in\Theta} \overline{U}(\theta) + \theta q.$$

In particular, this condition should hold for the equilibrium aggregate tariff $\overline{T}$. From (A2), it follows that $T^*(0) = 0$ and thus:

$$\text{(A3)} \quad \overline{T}(0) \leq 0.$$

Consider now the new tariff $\tilde{T}$ obtained from $\overline{T}$ as:

$$\tilde{T}(q) = \begin{cases} 0 & \text{if } q = 0, \\ \overline{T}(q) & \text{otherwise.} \end{cases}$$

Because of (A3), $\tilde{T}$ is itself upper semicontinuous. Moreover, we have:

$$\overline{T}(q) \leq \tilde{T}(q) \leq T^*(q) \quad \forall q \in \mathcal{Q}.$$

Because both $\overline{T}$ and T^* implement $(\overline{U}, \overline{q}, \overline{\delta})$, we deduce from those inequalities that $\tilde{T}$ also does so. Under $\tilde{T}$, every agent type chooses to participate $\tilde{\delta}(\theta) = 1$ because he has always the option to choose $q = 0$ and get thereby his reservation payoff that is normalized at zero. Because the objective function in (A1) has the same expected value at $(\overline{U}, \overline{q}, \overline{\delta})$ using $\overline{T}$ as it does at $(\overline{U}, \overline{q}, \tilde{\delta})$ using $\tilde{T}$, we conclude that

$$(\overline{U}, \overline{q}) \in \arg\max_{(U,q)\in\mathcal{I}} E_\theta \left[S(q(\theta)) - \theta q(\theta) - nU(\theta) + (n-1)(\tilde{T}(q(\theta)) - \theta q(\theta)) \right].$$

This new problem is again self-generating and $(\overline{U}, \overline{q})$ is equilibrium implemented by $\tilde{T}$.

Sufficiency. Consider a solution $(\overline{U}, \overline{q})$ to $(\overline{\mathcal{P}})$ which is implemented by the aggregate tariff $\overline{T}$. Note that because $(\overline{U}, \overline{q})$ is implemented by $\overline{T}$ with $\overline{T}(0) = 0$, we are considering the case where the agent always participates, $\delta = 1$. Construct individual tariffs $\overline{T}_i$ satisfying

$$\overline{T}_i(q) = S_i(q) - \frac{1}{n}(S(q) - \overline{T}(q)) \quad \forall i \in \{1, ..., n\}.$$

By construction,

$$\sum_{i=1}^{n} \overline{T}_i(q) = \overline{T}(q).$$

We show that this contract profile $(\overline{T}_1, ..., \overline{T}_n)$ is an equilibrium. Suppose indeed that all principals j for $j \neq i$ offer $\overline{T}_j$. At a best response, principal i induces an allocation (U, q, δ) that solves:

$$(\mathcal{P}_i): \quad \max_{(U,q,\delta)\in\mathcal{I}} E_\theta \left[S_i(q(\theta)) - \theta q(\theta) - U(\theta) + \overline{T}_{-i}(q(\theta)) \right].$$

Inserting the expressions of $\overline{T}_j$ (for $j \neq i$) using our construction above), the allocation that principal i would like to induce should solve

$$\max_{(U,q,\delta)\in\mathcal{I}} E_\theta \left[S(q(\theta)) - \theta q(\theta) - nU(\theta) + (n-1)(\overline{T}(q(\theta)) - \theta q) \right].$$

But this is the same maximization program in (A1), and hence principal i's choice $\overline{T}_i$ is a best response against $\overline{T}_{-i}$. *Q.E.D.*

Proof of Theorem 1: Necessity. Because of (3.1), any implementable rent profile is non-increasing. It follows that, necessarily, for any solution to $(\overline{\mathcal{P}})$ where the agent's rent is minimized, we must have:

(A4) $\quad \overline{U}(\theta_2) = 0.$

From (3.1) and (A4), we thus get (5.2). Inserting this expression of the rent into the maximand of $(\overline{\mathcal{P}})$ and integrating by parts shows that any solution to the relaxed problem

$(\overline{\mathcal{P}}^r)$ obtained when the convexity requirement on U has been omitted should also be a pointwise solution to:

$$\text{(A5)} \qquad \overline{q}(\theta) \in \arg\max_{q\in\mathcal{Q}} \; S(q) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right) q + (n-1)(\overline{T}(q) - \theta q), \text{ a.e.,}$$

where $\overline{T}$ implements $(\overline{U}, \overline{q})$.

Assumption 1 implies that $\theta + n\frac{F(\theta)}{f(\theta)}$ is increasing in θ. Therefore, it immediately follows from standard revealed preferences arguments that $\overline{q}(\theta)$ that solves (A5) is necessarily non-decreasing. Thus, the solution to the relaxed problem $(\overline{\mathcal{P}}^r)$ also solves $(\overline{\mathcal{P}})$ with the addition of the convexity requirement.

Define now the value function for the program (A5) as:

$$\text{(A6)} \qquad \overline{V}(\theta) \equiv \max_{q\in\mathcal{Q}} \; S(q) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right) q + (n-1)(\overline{T}(q) - \theta q).$$

Because the maximand on the right-hand side of (A6) is absolutely continuous in θ, upper semi-continuous in q and $\mathcal{Q}$ is compact, $V(\theta)$ is itself absolutely continuous. Moreover, given that $(\overline{U}, \overline{q})$ is an incentive-compatible allocation which solves this program, we have:

$$\overline{V}(\theta) = S(\overline{q}(\theta)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right) \overline{q}(\theta) + (n-1)\overline{U}(\theta).$$

Because $\overline{V}$ is absolutely continuous, it is almost everywhere differentiable and it satisfies the integral representation:

$$\overline{V}(\theta) - \overline{V}(\theta') = -\int_{\theta'}^{\theta} \left[\left(1 + n\frac{d}{d\theta}\left(\frac{F(\tilde{\theta})}{f(\tilde{\theta})}\right)\right)\overline{q}(\tilde{\theta}) + (n-1)\overline{q}(\tilde{\theta})\right] d\tilde{\theta} \quad \forall(\theta,\theta').$$

Because $\overline{U}$ is also absolutely continuous, we also have:

$$\overline{U}(\theta) - \overline{U}(\theta') = \int_{\theta}^{\theta'} \overline{q}(\tilde{\theta}) d\tilde{\theta} \quad \forall(\theta,\theta').$$

Note that

$$S(\overline{q}(\theta)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\theta) - \left[S(\overline{q}(\theta')) - \left(\theta' + n\frac{F(\theta')}{f(\theta')}\right)\overline{q}(\theta')\right]$$
$$= \overline{V}(\theta) - \overline{V}(\theta') - (n-1)\left[\overline{U}(\theta) - \overline{U}(\theta')\right].$$

Thus $S(\overline{q}(\theta)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\theta)$ is also absolutely continuous, and admits the following integral representation:

$$S(\overline{q}(\theta)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\theta) - \left[S(\overline{q}(\theta')) - \left(\theta' + n\frac{F(\theta')}{f(\theta')}\right)\overline{q}(\theta')\right]$$
$$= -\int_{\theta'}^{\theta} \left(1 + n\frac{d}{d\theta}\left(\frac{F(\tilde{\theta})}{f(\tilde{\theta})}\right)\right)\overline{q}(\tilde{\theta})d\tilde{\theta}.$$

Observe that

$$\left(\frac{F(\theta)}{f(\theta)} - \frac{F(\theta')}{f(\theta')}\right)\overline{q}(\theta') = \int_{\theta'}^{\theta} \frac{d}{d\theta}\left(\frac{F(\tilde{\theta})}{f(\tilde{\theta})}\right)\overline{q}(\theta')d\tilde{\theta}.$$

Using this latter condition, Assumption 1 and the fact that $\overline{q}$ is non-increasing, we obtain:

$$S(\overline{q}(\theta)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\theta) - \left[S(\overline{q}(\theta')) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\theta')\right]$$

$$= \int_{\theta'}^{\theta}\left[1 + n\frac{d}{d\theta}\left(\frac{F(\tilde{\theta})}{f(\tilde{\theta})}\right)\right](\overline{q}(\tilde{\theta}) - \overline{q}(\theta'))d\tilde{\theta} \geq 0.$$

Because any $q' \in \overline{q}(\Theta)$ can be identified with some $\theta' \in \Theta$, the inequality implies $\overline{q}(\theta)$ satisfies (5.1) pointwise in θ.

SUFFICIENCY. Consider any allocation $(\overline{U}, \overline{q})$ that satisfies (5.1), and (5.2). Simple revealed preferences arguments from (5.1) together with Assumption 1 imply that $\overline{q}$ is non-increasing. Then, (5.2) implies that $\overline{U}$ is convex. We now prove that this allocation is equilibrium implementable. First, we construct an aggregate transfer by duality as in (3.3) and obtain $\overline{T}$ from (3.6). By construction $\overline{q}(\theta)$ is a maximizer of $\overline{T}(q) - \theta q$ over $\overline{q}(\Theta)$. Second, individual contracts are then recovered by using (4.1). Third, we need to check that the optimality conditions (A5) for the surrogate principal's optimization problem are satisfied. This last step is an immediate consequence from the fact that $\overline{q}(\theta)$ is a maximizer for both $S(q) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)q$ and $\overline{T}(q) - \theta q$ over $\overline{q}(\Theta)$ so that it also maximizes a convex combination of both objectives. *Q.E.D.*

PROOF OF PROPOSITION 2: Define $T^m(q) = \min_{q \in \mathcal{Q}} U^m(\theta) + \theta q$. This allows us to define a self-generating problem associated to that tariff and to check that (U^m, q^m) is a solution. Indeed, once one has taken care of the expression of the rent and integrating by parts, this self-generating problem can be rewritten as in (A5) in terms of output only as:

$$\text{(A7)} \qquad q^m(\theta) \in \arg\max_{q \in \mathcal{Q}} \; S(q) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)q + (n-1)(T^m(q) - \theta q).$$

That q^m is indeed a solution then follows from the fact that $q^m(\theta) \in \arg\max_{q \in \mathcal{Q}} \; S(q) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)q$ and $q^m(\theta) \in \arg\max_{q \in \mathcal{Q}} \; (n-1)(T^m(q) - \theta q)$. *Q.E.D.*

PROOF OF PROPOSITION 3: NECESSITY. That $\overline{q}$ should be non-increasing follows from the first step in the proof of Theorem 1. Thus, $\overline{q}$ is almost everywhere differentiable. At any point of differentiability, the first-order necessary condition for optimality of the incentive problem

$$\theta \in \arg\max_{\hat{\theta} \in \Theta} \; S(\overline{q}(\hat{\theta})) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\hat{\theta}) \quad \forall \theta \in \Theta$$

gives us (5.6).

Consider now the value function V as defined in (A6). V is continuous and thus at a point of discontinuity θ_0 for $\overline{q}$ we have:

$$\text{(A8)}\qquad \lim_{\theta\to\theta_0^-} V(\theta) = \lim_{\theta\to\theta_0^+} V(\theta).$$

Because $\overline{q}$ is non-increasing, it is almost everywhere differentiable and such points of discontinuity are necessarily isolated. On the right- and the left- hand neighborhoods of θ_0, (5.6) thus applies and either $\dot{\overline{q}}(\theta) = 0$ or $\overline{q}(\theta) = q^m(\theta)$. Suppose for instance that bunching arises on the left-neighborhood only and call thus $\overline{q}(\theta_0^-)$ this left-hand side value of $\overline{q}$ with $\overline{q}(\theta_0^-) > q^m(\theta_0)$ because $\overline{q}$ cannot be non-decreasing at such discontinuity. Because the agent's rent $\overline{U}$ is also continuous at θ_0, we have:

$$\text{(A9)}\qquad \lim_{\theta\to\theta_0^-} \overline{T}(\overline{q}(\theta_0^-)) - \theta\overline{q}(\theta_0^-) = \lim_{\theta\to\theta_0^+} \overline{T}(q^m(\theta_0)) - \theta q^m(\theta_0).$$

Inserting into (A8) and simplifying yields:

$$\lim_{\theta\to\theta_0^-} S(\overline{q}(\theta_0^-)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\theta_0^-) = \lim_{\theta\to\theta_0^+} S(q^m(\theta)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right) q^m(\theta).$$

Expressing those right- and left-hand side limits gives us

$$\text{(A10)}\quad S(\overline{q}(\theta_0^-)) - \left(\theta_0 + n\frac{F(\theta_0)}{f(\theta_0)}\right)\overline{q}(\theta_0^-) = S(q^m(\theta_0)) - \left(\theta_0 + n\frac{F(\theta_0)}{f(\theta_0)}\right) q^m(\theta_0).$$

By definition, $q^m(\theta_0)$ achieves the maximum of $S(q) - \left(\theta_0 + n\frac{F(\theta_0)}{f(\theta_0)}\right) q$ and (A10) necessarily implies that $\overline{q}(\theta_0^-) = q^m(\theta_0)$. A contradiction.

Similarly, we could also rule out the case where bunching only arises on the right-neighborhood of θ_0 at a value $\overline{q}(\theta_0^+)$. Hence, the only configuration that is consistent with a discontinuity at an interior θ_0 is to have $\dot{\overline{q}}(\theta) = 0$ and thus bunching on both sides of such discontinuity.

Taking stock of this finding, inserting again (A10) into (A8) and simplifying now yields:

$$\lim_{\theta\to\theta_0^-} S(\overline{q}(\theta_0^-)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\theta_0^-) = \lim_{\theta\to\theta_0^+} S(\overline{q}(\theta_0^+)) - \left(\theta + n\frac{F(\theta)}{f(\theta)}\right)\overline{q}(\theta_0^+).$$

Expressing the right- and left-hand side limits finally gives us (5.7).

Sufficiency. It directly follows from the proof of Theorem 1. *Q.E.D.*

Proof of Proposition 4: From the text, we know that the maximal equilibrium (U^m, q^m) is a biconjugate equilibrium. Consider another equilibrium $(\overline{U}, \overline{q})$ with at least one discontinuity hole $(q^m(\theta_0^+), q^m(\theta_0^-))$ for $\theta_0 \in (\theta_1, \theta_2)$. The only thing to prove is that replacing the implementing tariff $\overline{T}$ as proposed in (3.6) by the least-concave tariff T^*

changes the solution to the self-generating problem $(\overline{\mathcal{P}})$ and that $(\overline{U}, \overline{q})$ is not a solution to the new self-generating problem $(\overline{\mathcal{P}}^*)$ so constructed. From a remark in the text, T^* is concave, and linear over $(q^m(\theta_0^+), q^m(\theta_0^-))$ with slope θ_0. The surrogate surplus at θ_0 is itself concave

$$S(q) - \left(\theta_0 + n\frac{F(\theta_0)}{f(\theta_0)}\right) q + (n-1)(\overline{T}(q) - \theta_0 q)$$

and maximized at $q^m(\theta_0)$; a contradiction with the fact that, at $(\overline{U}, \overline{q})$, θ_0 is indifferent between $q^m(\theta_0^+)$ and $q^m(\theta_0^-)$. *Q.E.D.*

Proof of Proposition 5: We make use of Proposition 1 in Amador and Bagwell (2013). Using the notation of Amador and Bagwell (2013), we transform θ into the type, $\gamma = -(\theta + nF(\theta)/f(\theta))$, and let $\phi(\gamma)$ denote the inverse mapping. The corresponding density for γ is given by $\tilde{f}(\gamma) = f(\phi(\gamma))|\phi'(\gamma)|$. Let $\tilde{F}(\gamma)$ denote the distribution of γ. We transform output q into the choice π and define the overall surplus as $b(\pi) = S(\pi)$ so that the agent's preferences can be written as

$$b(\pi) + \gamma\pi$$

as required. The principal's preferences can now be also rewritten as

$$w(\gamma, \pi) = b(\pi) + \gamma\pi + (n-1)F(\phi(\gamma))/f(\phi(\gamma))\pi.$$

Because the agent's best action, $\pi_f(\gamma) = q^m(\phi(\gamma))$, is strictly positive for all γ, it is straightforward to check that our setting satisfies Assumption 1 in Amador and Bagwell (2013).

To prove our interval set is the optimal form of delegation, we need to verify conditions (c1), (c2), (c2'), (c3) and (c3') of Amador and Bagwell (2013). Given our conjectured solution, conditions (c2) and(c3') are satisfied. Given the principal and agent have perfectly aligned preferences for $\gamma = \phi(\theta_1)$, condition (c2') is satisfied. Assuming that condition (c1) is satisfied and that $\gamma^L = \phi(\hat{\theta})$, where $\hat{\theta}$ is defined in (6.1), then condition (c3) is also satisfied. Condition (6.1) is the defining first-order condition for finding an optimal floor delegation set. What remains to be show is that (c1) is satisfied. Defining $H(\theta) = F(\theta)/f(\theta)$, it is sufficient for (c1) that

$$\tilde{F}(\gamma) - (n-1)H(\phi(\gamma))\tilde{f}(\gamma)$$

is nondecreasing. Differentiating, we shall prove

$$\tilde{f}(\gamma) - (n-1)\left(H'(\phi(\gamma))\tilde{f}(\gamma)\phi'(\gamma) + H(\phi(\gamma))\tilde{f}'(\gamma)\right) \geq 0.$$

Using $\tilde{f} = -f\phi'$, we have $\tilde{f}'(\gamma) = -f'(\theta)\phi'^2 - f(\theta)\phi''(\gamma)$. Because

$$\phi'(\gamma) = \frac{-1}{1 + nH'(\phi(\gamma))},$$

we have $\phi''(\gamma) = \phi'^3 nH''(\phi(\gamma))$. Substituting these relationships into our inequality and simplifying, we reduce our required condition to Assumption 2.

Consider thus a *"floor equilibrium"* with an allocation

(A11) $\overline{q}(\theta) = \max\{q^m(\theta), q^m(\theta^*)\}$ for some $\theta^* \in \Theta$.

together with the rent minimization requirement (5.2). The maximal equilibrium is simply obtained by choosing $\theta^* = \theta_2$ as a special case. The sum of the principal's profits evaluated at such equilibria can be expressed in terms of the floor θ^* only as:

$$V(\theta^*) = \int_{\theta_1}^{\theta^*} \left(S(q^m(\theta)) - \left(\theta + \frac{F(\theta)}{f(\theta)}\right) q^m(\theta) \right) f(\theta) d\theta$$

$$+ \int_{\theta^*}^{\theta_2} \left(S(q^m(\theta^*)) - \left(\theta + \frac{F(\theta)}{f(\theta)}\right) q^m(\theta^*) \right) f(\theta) d\theta.$$

Differentiating with respect to θ^* yields:

$$\dot{V}(\theta^*) = \dot{q}^m(\theta^*) \int_{\theta^*}^{\theta_2} \left(S'(q^m(\theta^*)) - \theta - \frac{F(\theta)}{f(\theta)} \right) f(\theta) d\theta.$$

From this, we immediately get that:

(A12) $\dot{V}(\theta_1) = \dot{q}^m(\theta_1) \int_{\theta_1}^{\theta_2} \left(\theta_1 - \theta - \frac{F(\theta)}{f(\theta)} \right) f(\theta) d\theta > 0,$

where the last inequality follows from $\theta_1 - \theta - \frac{F(\theta)}{f(\theta)} < 0$ and $\dot{q}^m(\theta_1) < 0$. Second, we also obtain:

(A13) $\dot{V}(\theta_2) = 0$ with $\ddot{V}(\theta_2) = -(n-1)\dot{q}^m(\theta_2) > 0$

when Assumption 1 holds and $q^m(\theta_2) > 0$ so that $q^m(\theta)$ is strictly decreasing in that neighborhood of θ_2. Hence, although θ_2 is a local extremum of V, it corresponds to a minimum. It follows that the maximal equilibrium is never optimal. From (A12), we derive the existence of a maximum $\hat{\theta}$ which is necessarily interior. Rewriting the condition $\dot{V}(\hat{\theta}) = 0$ gives us (6.1). *Q.E.D.*

Proof of Theorem 2: Necessity. The agent's incentive compatibility conditions imply that any equilibrium allocation must satisfy the monotonicity condition

(A14) $\overline{q}(\theta_1) \geq \overline{q}(\theta_2)$.

Assuming a discrete distribution of types, we consider now the relaxed self-generating program $(\overline{\mathcal{P}}^r)$ obtained from $(\overline{\mathcal{P}})$ by ignoring (A14) and focusing only on the low-cost type's incentive constraint and the high-cost type's participation constraint:

(A15) $U(\theta_1) \geq U(\theta_2) + \Delta\theta q(\overline{\theta})$,

(A16) $U(\theta_2) \geq 0$.

At the optimum of $(\overline{\mathcal{P}}^r)$, (A15) and (A16) are both binding. Hence, the equilibrium profile of rents $\overline{U}$ must satisfy (7.4). Inserting the expressions of $U(\theta)$ obtained from (A15) and

(A16) binding into the maximand of $(\overline{\mathcal{P}}^r)$, this maximand can be rewritten in terms of outputs only as:

$$\text{(A17)}\quad E_\theta\left[S(q(\theta)) - \theta q(\theta) + (n-1)(\overline{T}(q(\theta)) - \theta q(\theta))\right] - n\Delta\theta q(\theta_2).$$

By definition of an equilibrium, the non-increasing output schedule $(\overline{q}(\theta_1), \overline{q}(\theta_2))$ maximizes this expression over the set of such non-increasing output schedule. Thus, $(\overline{q}(\theta_1), \overline{q}(\theta_2))$ is weakly preferred to any other non-increasing pair $(q(\theta_1, q(\theta_2))$. It imposes a number of requirements that we now describe.

- Because $(\overline{q}(\theta_1), \overline{q}(\theta_2))$ is weakly preferred to $(\overline{q}(\theta_2), \overline{q}(\theta_2))$, we have

$$\text{(A18)}\quad S(\overline{q}(\theta_1)) - \theta_1\overline{q}(\theta_1) + (n-1)(\overline{T}(\overline{q}(\theta_1)) - \theta_1\overline{q}(\theta_1))$$
$$\geq S(\overline{q}(\theta_2)) - \theta_1\overline{q}(\theta_2) + (n-1)(\overline{T}(\overline{q}(\theta_2)) - \theta_1\overline{q}(\theta_2)).$$

Since $\overline{T}$ implements $(\overline{U}, \overline{q})$ and (7.4) holds for this allocation, we obtain

$$\overline{T}(\overline{q}(\theta_1)) - \theta_1\overline{q}(\theta_1) = \overline{T}(\overline{q}(\theta_2)) - \theta_1\overline{q}(\theta_2).$$

Inserting into (A18) and simplifying immediately gives (7.2).

- Because $(\overline{q}(\theta_1), \overline{q}(\theta_2))$ is weakly preferred to $(\overline{q}(\theta_1), \overline{q}(\theta_1))$, we have

$$\text{(A19)}\quad S(\overline{q}(\theta_2)) - \left(\theta_2 + \frac{n\nu}{1-\nu}\Delta\theta\right)\overline{q}(\theta_2) + (n-1)(\overline{T}(\overline{q}(\theta_2)) - \theta_2\overline{q}(\theta_2)))$$
$$\geq S(\overline{q}(\theta_2)) - \left(\theta_2 + \frac{n\nu}{1-\nu}\Delta\theta\right)\overline{q}(\theta_1) + (n-1)(\overline{T}(\overline{q}(\theta_1)) - \theta_2\overline{q}(\theta_2)).$$

Since $\overline{T}$ implements the equilibrium allocation $(\overline{U}, \overline{q})$ and (7.4) holds for this allocation, we obtain

$$\text{(A20)}\quad \overline{T}(\overline{q}(\theta_2)) - \theta_2\overline{q}(\theta_2) = 0 \text{ and } \overline{T}(\overline{q}(\theta_1)) - \theta_2\overline{q}(\theta_1) = -\Delta\theta(\overline{q}(\theta_1) - \overline{q}(\theta_1)).$$

Inserting into (A19) and simplifying immediately gives the first condition in (7.3).

- Because $(\overline{q}(\theta_1), \overline{q}(\theta_2))$ is weakly preferred to $(\overline{q}(\theta_1), 0)$, we have

$$\text{(A21)}\quad S(\overline{q}(\theta_2)) - \left(\theta_2 + \frac{n\nu}{1-\nu}\Delta\theta\right)\overline{q}(\theta_2) + (n-1)(\overline{T}(\overline{q}(\theta_2)) - \theta_2\overline{q}(\theta_2))) \geq 0,$$

where the right-hand side above uses the normalization $S(0) = 0$. Using now the first condition in (A20) and simplifying gives us the second condition in (7.3).

- Finally, summing together (7.2) and the first condition in (7.3) immediately gives us that any allocation that solves the relaxed problem $(\overline{\mathcal{P}}^r)$ also satisfies (A14) and thus solves $(\overline{\mathcal{P}})$.

Sufficiency. Take any allocation $(\overline{U}, \overline{q})$ satisfying constraints (7.2), (7.3) (7.4) and (A14). We check that such allocation solves $(\overline{\mathcal{P}})$ for an aggregate $\overline{T}$ that implements the allocation.

First, observe that for such allocation, (7.2) and (7.3) taken together implies:

$$S(\overline{q}(\theta_1)) - \theta_1\overline{q}(\theta_1) \geq S(\overline{q}(\theta_1)) - \theta_2\overline{q}(\theta_1) + \Delta\theta\overline{q}(\theta_2) \geq \left(\frac{n\nu}{1-\nu} + 1\right)\Delta\theta\overline{q}(\theta_2) \geq 0.$$

From this, together with (7.3) and (7.4), we deduce that the objective function in $(\overline{\mathcal{P}})$ is necessarily positive for an allocation $(\overline{U}, \overline{q})$ that is implemented by a contract $\overline{T}$.

Consider thus the aggregate $\overline{T}$ defined in (7.6). Two facts follow from this definition. First, $\overline{T}$ implements $(\overline{U}, \overline{q})$ and, in particular, (7.4) holds. Second,choosing any implementable output profile $q(\theta_1) \geq q(\theta_2)$ other than $(\overline{q}(\theta_1), \overline{q}(\theta_2))$, $(\overline{q}(\theta_2), \overline{q}(\theta_2))$, $(\overline{q}(\theta_1), \overline{q}(\theta_1))$, $(\overline{q}(\theta_1), 0)$ or $(0,0)$ leads to an infinitively negative payoff for the surrogate principal. Thus the surrogate principal who wants maximizes $(\overline{\mathcal{P}})$ can restrict himself to choose within those allocations, since only these ones lead to a non-negative payoff. Conditions (7.2) and (7.3) then ensure that the best such profile is indeed $(\overline{q}(\theta_1), \overline{q}(\theta_2))$.

From those two facts, it also follows that $(\overline{U}, \overline{q})$ solves $(\overline{\mathcal{P}})$. Using the construction (4.1) allows us to retrieve all individual equilibrium offers $\overline{T}_i$. *Q.E.D.*

Proof of Theorem 3: From Proposition 1, an allocation $(\overline{U}, \overline{q})$ is equilibrium implementable if and only if it solves problem $(\overline{\mathcal{P}})$ where $\overline{T}$ is an aggregate tariff that implements $(\overline{U}, \overline{q})$. From the proof of Theorem 2, the solution to $(\overline{\mathcal{P}})$ actually solves the relaxed problem $(\overline{\mathcal{P}}^r)$. Inserting the expression of the agent's rent in terms of outputs, solving $(\overline{\mathcal{P}})$ amounts to again maximizing (A17) with respect to $(q(\theta_1), q(\theta_2))$. Because we are now restricting attention to equilibrium allocations supported by a *biconjugate* aggregate schedule $\overline{T}$, maximizing (A17) is a concave program whose solution is characterized by generalized first-order conditions using subdifferentials. Inserting the values of those subdifferentials into the optimality conditions for the maximization of (A17) by means of subdifferentials yields (7.9) (resp. (7.10)) in state θ_1 (resp. θ_2). It is straightforward to check that the aggregate contract defined in (7.8) is non-decreasing and concave. Sufficiency follows along the same lines as in the proof of Theorem 2. *Q.E.D.*

Proof of Proposition 6: Since (7.4) holds for allocations in $\mathcal{I}^{eq}$, the principals' *ex ante* maximization problem consists in maximizing

(A22) $\quad E_\theta\left[S(q(\theta)) - \theta q(\theta)\right] - n\Delta\theta q(\theta_2)$

subject to (7.2) and (7.3). The optimal output for the low-cost agent is thus first-best. When Assumption 3 holds, the cooperative outcome $q^c(\theta_2)$ is not implementable and the best output for a high-cost agent is obtained when (7.3) is binding. *Q.E.D.*

Proof of Proposition 7: From Theorem 3, the principals' *ex ante* maximization problem consists in maximizing (A22) subject to $\overline{q}(\theta_1) = q^{fb}(\theta_1)$ and $\overline{q}(\theta_2) \in [\tilde{q}(\theta_2,), q^*(\theta_2)]$ The maximum is achieved at $q^*(\theta_2)$ since $q^*(\theta_2) < q^c(\theta_2)$. *Q.E.D.*

Proof of Theorem 4: As in the proof of Theorem 1, (5.2) holds for any solution to the self-generating problem $(\overline{\mathcal{P}}_k)$ corresponding to a distribution F_k defined in (7.12). Inserting (5.2) into the maximand of $(\overline{\mathcal{P}}_k)$ and integrating by parts again shows that any solution to the corresponding relaxed problems $(\overline{\mathcal{P}}_k^r)$ obtained when the convexity requirement has been omitted should again be a pointwise solution to (A5) with the specific choice of the distribution (resp. density) function F_k (resp. f_k).

Condition (7.13) implies that, for k large enough, $\theta + n\frac{F_k(\theta)}{f_k(\theta)}$ is decreasing in θ for θ close enough to θ_2. Therefore, the maximal output defined as:

$$\text{(A23)} \quad S'(q_k^m(\theta)) = \theta + n\frac{F_k(\theta)}{f_k(\theta)} \quad \forall \theta \in \Theta$$

cannot be everywhere non-increasing and thus fails to be an equilibrium. Bunching always arises on the upper tail of the distribution. Any sequence of equilibrium allocations $(\overline{U}_k, \overline{q}_k)$ which is equilibrium implemented by tariffs $\overline{T}_k$ features bunching on an interval $[\theta_k^*, \theta_2]$ for k large enough.

Because the corresponding allocations must solve the self-generating problem $(\overline{\mathcal{P}}_k)$ constructed from the corresponding $\overline{T}_k$, these equilibria satisfy the optimality condition:

$$\overline{q}_k \in \arg\max_{q(\theta)\in\mathcal{Q}} E_\theta\left(S(q(\theta)) - \left(\theta + n\frac{F_k(\theta)}{f_k(\theta)}\right)q(\theta) + (n-1)(\overline{T}_k(q(\theta)) - \theta q(\theta))\right)$$

subject to $q(\theta)$ non-increasing.

Following Myerson (1981), we first define for any $u \in [0, 1]$

$$h_k(u) = z_k(F_k^{-1}(u)) \text{ and } H_k(u) = \int_0^u h_k(\tilde{u})d\tilde{u},$$

where F_k^{-1} exists thanks to Assumption 4. Let $G_k = covH_k$ be the convex hull of H_k. Since G_k is convex, it is almost everywhere differentiable with a non-decreasing derivative g_k. Define accordingly the monotonic function:

$$\overline{z}_k(\theta) = g_k(F(\theta)).$$

It is well known that $\overline{z}_k(\theta) = z_k(\theta)$ when $\overline{z}_k(\theta)$ is increasing. More specifically to our context and our choice of an hill-shaped hazard rate, it can be readily shown that

$$\overline{z}_k(\theta) = \overline{z}_k(\theta_k^*) \quad \forall \theta \in [\theta_k^*, \theta_2],$$

where $\theta_k^* \in (\theta_1, \theta_2)$ is defined as

$$\text{(A24)} \quad z_k(\theta_k^*) = E_\theta(z_k(\theta|\theta \geq \theta_k^*) \Leftrightarrow \frac{F_k(\theta_k^*)}{f_k(\theta_k^*)} = \frac{\theta_2 - \theta_k^*}{1 - F_k(\theta_k^*)}.$$

The function $\overline{z}_k(\theta)$ can be viewed as an *"ironed version"* of the virtual cost $z_k(\theta)$. Following the same steps as in Myerson (1981, p. 69), the solution to the following maximization (where the integrand has been ironed) is non-increasing and it is thus also a solution to the original problem:

$$\text{(A25)} \quad \overline{q}_k(\theta) \in \arg\max_{q\in\mathcal{Q}} S(q) + (n-1)\overline{T}_k(q) - n\overline{z}_k(\theta)q \quad \forall \theta \in \Theta.$$

From the agent's incentive compatibility constraints, we may also rewrite:

$$\text{(A26)} \quad \overline{q}_k(\theta) \in \arg\max_{q\in\mathcal{Q}} \overline{T}_k(q) - \theta q \quad \forall \theta \in \Theta.$$

By construction, $\overline{z}_k$ is non-decreasing. Let also define

$$\hat{V}(\overline{z}_k) = \max_{q\in\mathcal{Q}} S(q) + (n-1)\overline{T}_k(q) - n\overline{z}_k q \quad \forall \overline{z}_k.$$

By a familiar conjugacy argument, we may also define:

$$\tilde{V}_k(q) = \min_{\overline{z}_k} \hat{V}(\overline{z}_k) + n\overline{z}_k q.$$

$\tilde{V}_k$ is a minimum of linear functions and as such is concave in q From this definition, it also follows that any equilibrium aggregate tariff $\overline{T}_k$ must satisfy:

$$\text{(A27)} \quad (n-1)\overline{T}_k(q) \le (n-1)\overline{T}^*_k(q) = \tilde{V}_k(q) - S(q)$$

with equality at any $q \in \overline{q}_k(\Theta)$. Because $(n-1)\overline{T}^*_k(q) + S(q) = \tilde{V}_k(q)$ is concave in q, it admits a subdifferential $\partial\tilde{V}_k$ at any equilibrium point $\overline{q}_k(\theta)$. From this, we get that for any $s \in \partial\tilde{V}_k(\overline{q}_k(\theta))$:

$$S(q) + (n-1)\overline{T}^*_k(q) \le S(\overline{q}_k(\theta)) + (n-1)\overline{T}^*_k(\overline{q}_k(\theta)) + s(q - \overline{q}_k(\theta)) \quad \forall q \in \mathcal{Q}.$$

Using (A27), we obtain:

$$S(q) + (n-1)\overline{T}_k(q) \le S(\overline{q}_k(\theta)) + (n-1)\overline{T}^*_k(\overline{q}_k(\theta)) + s(q - \overline{q}_k(\theta)) \quad \forall q \in \mathcal{Q}.$$

Because (A27) holds as an equality at $q \in \overline{q}_k(\Theta)$, we have also:

$$\text{(A28)} \quad S(q) + (n-1)\overline{T}_k(q) \le S(\overline{q}_k(\theta)) + (n-1)\overline{T}_k(\overline{q}_k(\theta)) + s(q - \overline{q}_k(\theta)) \quad \forall q \in \mathcal{Q}.$$

Because S is twice continuously differentiable and $\mathcal{Q}$ is compact, S'' is bounded above and there exists M such that:

$$\text{(A29)} \quad S(\overline{q}_k(\theta)) - S(q) - S'(\overline{q}_k(\theta))(\overline{q}_k(\theta) - q) \le \frac{M}{2}(\overline{q}_k(\theta) - q)^2. \quad \forall q \in \mathcal{Q}.$$

Inserting into (A28) yields:

$$(n-1)\overline{T}_k(q) \le (n-1)\overline{T}^*_k(\overline{q}_k(\theta)) + (s + S'(\overline{q}_k(\theta)))(q - \overline{q}_k(\theta)) + \frac{M}{2}(\overline{q}_k(\theta) - q)^2 \quad \forall q \in \mathcal{Q}.$$

In other words, $(n-1)\overline{T}_k(q)$ admits a proximal subdifferential at $q = \overline{q}_k(\theta)$,[20] denoted by $\partial^p\overline{T}_k(\overline{q}_k(\theta))$, and we have the following inclusion:

$$S'(\overline{q}_k(\theta)) + \partial V(\overline{q}_k(\theta)) \subseteq (n-1)\partial^p\overline{T}_k(\overline{q}_k(\theta)) \quad \forall \theta \in \Theta.$$

Since $\overline{T}_k(q)$ admits a proximal subdifferential at $q = \overline{q}_k(\theta)$, the agent's incentive compatibility condition (A26) can be written as:[21]

$$\theta \in \partial^p\overline{T}_k(\overline{q}_k(\theta)) \quad \forall \theta \in \Theta.$$

[20] Vinter (2000, p. 41).
[21] Vinter (2000, p. 45).

From which, we derive the following inclusion:

$$\vartheta_k(\overline{q}_k(\theta)) \subseteq \partial^p \overline{T}_k(\overline{q}_k(\theta)) \quad \forall \theta \in \Theta,$$

where $\vartheta_k(q)$ is the inverse correspondence of $\overline{q}_k$. By incentive compatibility, we also know that, for $\theta' \notin \vartheta_k(\overline{q}_k(\theta))$, we have:

$$\overline{T}_k(\overline{q}_k(\theta)) - \theta'\overline{q}_k(\theta) < \overline{T}_k(\overline{q}_k(\theta')) - \theta'\overline{q}_k(\theta')$$

and thus $\theta'^p \overline{T}_k(\overline{q}_k(\theta))$ which proves the reverse inclusion

$$\partial^p \overline{T}_k(\overline{q}_k(\theta)) \subseteq \vartheta_k(\overline{q}_k(\theta)) \quad \forall \theta \in \Theta.$$

Thus, the following equality holds:

$$\text{(A30)} \quad \partial^p \overline{T}_k(\overline{q}_k(\theta)) = \vartheta_k(\overline{q}_k(\theta)) \quad \forall \theta \in \Theta.$$

Turning now to (A25), we rewrite this condition as:

$$S(q) + (n-1)\overline{T}_k(q) - n\overline{z}_k(\theta)q \leq S(\overline{q}_k(\theta)) + (n-1)\overline{T}_k(\overline{q}_k(\theta)) - n\overline{z}_k(\theta)\overline{q}_k(\theta)$$

or, using (A29)

$$(nz_k(\theta) - S'(\overline{q}_k(\theta)))(\overline{q}_k(\theta) - q) \leq (n-1)(\overline{T}_k(\overline{q}_k(\theta)) - \overline{T}_k(q)) + \frac{M}{2}(\overline{q}_k(\theta) - q)^2.$$

In other words, we get:

$$nz_k(\theta) - S'(\overline{q}_k(\theta)) \in (n-1)\partial^p \overline{T}_k(\overline{q}_k(\theta)).$$

and thus

$$\text{(A31)} \quad 0 \in S'(\overline{q}_k(\theta)) + (n-1)\partial^p \overline{T}_k(\overline{q}_k(\theta)) - n\overline{z}_k(\theta) \quad \forall \theta \in \Theta.$$

We draw two implications from those findings.

- Condition (A31) taken for the bunching output $\overline{q}_k^p$ on $[\theta_k^*, \theta_2]$ gives us:

  $$0 \in S'(\overline{q}_k^p) + (n-1)\partial^p \overline{T}_k(\overline{q}_k^p) - n\overline{z}_k(\theta_k^*).$$

  Taking into account that $\partial^p \overline{T}_k(\overline{q}_k^p) = [\theta_k^*, \theta_2]$ and inserting into (A31) yields that, for k large enough, any equilibrium allocation $(\overline{U}_k, \overline{q}_k)$ entails bunching on an interval $[\theta_k^*, \theta_2]$ at an output $\overline{q}_k^p$ such that

  $$\text{(A32)} \quad 0 \in S'(\overline{q}_k^p) + (n-1)\overline{\vartheta}_k(\overline{q}_k^p) - n\overline{z}_k(\theta_k^*),$$

  where $\overline{\vartheta}_k(\overline{q}_k^p) = [\theta_k^*, \theta_2]$ and $\lim_{k \to +\infty} \theta_k^* = \theta_1$.
- On the interval $[\theta_1, \theta_k^*]$, $\overline{z}_k(\theta)$ is increasing. The analysis of Theorem 1 and Proposition 3 applies on this interval *mutatis mutandis*. In particular, all equilibria satisfy almost everywhere on this interval the conditions:
  1. $\overline{q}_k$ is non-increasing,
  2. At any point θ of differentiability $\overline{q}_k$, i.e., almost everywhere, the following condition holds:

     $$\text{(A33)} \quad \dot{\overline{q}}_k(\theta)\left(S'(\overline{q}_k(\theta)) - \theta - n\frac{F_k(\theta)}{f_k(\theta)}\right) = 0,$$

3. At any interior point of discontinuity $\theta_0 \in (\theta_1, \theta_k^*)$, $\overline{q}_k$ is right- and left-continuous and the surrogate surplus remains continuous:

(A34)
$$S(\overline{q}_k(\theta_0^+)) - \left(\theta_0 + n\frac{F_k(\theta_0)}{f_k(\theta_0)}\right)\overline{q}_k(\theta_0^+) = S(\overline{q}_k(\theta_0^-)) - \left(\theta_0 + n\frac{F_k(\theta_0)}{f_k(\theta_0)}\right)\overline{q}_k(\theta_0^-).$$

In particular, any equilibrium with a separating solution on a right-neighborhood of θ_1, say $[\theta_1, \tilde{\theta}_k^*]$, is such that $\partial^p \overline{T}_k$ is single-valued on that right-neighborhood with

$$\partial^p \overline{T}(\overline{q}_k(\theta)) = \theta \quad \forall \theta \in [\theta_1, \tilde{\theta}_k^*].$$

Inserting into (A31) gives us immediately

$$\overline{q}_k(\theta) = q_k^m(\theta) \quad \forall \theta \in [\theta_1, \tilde{\theta}_k^*].$$

In particular, we get a familiar *"no distortion at the top"* result:

(A35) $$\overline{q}_k(\theta_1) = q^{fb}(\theta_1), \quad \forall k \in \mathbb{N}.$$

Take now any sequence of equilibrium allocations $(\overline{U}_k, \overline{q}_k)$ with an output profile $\overline{q}_k$ which remains separating on a right-neighborhood of θ_1 and that converges towards an allocation $(\overline{U}_\infty, \overline{q}_\infty)$. Three facts follow from our previous analysis for such limit.

- Passing to the limit in (A35) preserves the *"no distortion at the top"* result:

$$\overline{q}_\infty(\theta_1) = q^{fb}(\theta_1).$$

- When k is large enough, $F_k(\theta)$ converges towards ν for all $\theta \in [\theta_k^*, \theta_2)$ by Assumption 4. Inserting into (A24) gives the following approximation:

$$\frac{F_k(\theta_k^*)}{f_k(\theta_k^*)} \approx \frac{\theta_2 - \theta_k^*}{1-\nu}.$$

Because Θ is compact, the sequence θ_k^* has converging subsequences. Call θ_∞ any such limit. By Assumption 4 again, the left-hand side converges towards ∞ along such subsequence unless $\theta_\infty = \underline{\theta}_1$. The only possibility is thus that all subsequences and thus the sequence θ_k^* itself converge so that:

(A36) $$\lim_{k \to +\infty} \theta_k^* = \theta_1 \text{ and } \lim_{k \to +\infty} z_k(\theta_k^*) = \theta_1 + \frac{\Delta\theta}{1-\nu}.$$

- Inserting (A36) into (A32) yields thus

$$0 \in S'(\overline{q}_\infty^p) + (n-1)\,[\theta_1, \theta_2] - n\left(\theta_1 + \frac{\Delta\theta}{1-\nu}\right)$$

which amounts to

$$\overline{q}_\infty^p \in [\tilde{q}(\theta_2), q^*(\theta_2))].$$

Hence, (7.14) holds and any sequence $\overline{q}_k$ of output equilibrium profile converges pointwise towards the output allocation in a *biconjugate* equilibrium. *Q.E.D.*