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Existence and continuity of the optimal contract in adverse selection models with constraints[*]

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Abstract

We consider a general adverse selection model as an optimal control problem with mixed constraints. We prove that under broad conditions the optimal contract exists and is continuous.

Key words: *Mixed constraints, optimal control, adverse selection*
JEL Classification: C61, C62, D82, D86.

Résumé

Nous considérons un modèle général de sélection adverse comme un problème de contrôle optimal avec contraintes mixtes. Nous prouvons que sous conditions globales le contrat optimal existe et est continu.

Mots clés : *Contraintes mixtes, contrôle optimal, sélection adverse*
Classification JEL : C61, C62, D82, D86.

Introduction

The theory of incentives, particularly the theory of adverse selection, is normally silent when it comes to the existence and continuity of the optimal contracts. Usually the existence and continuity of the optimal contract are taken as given. This implies that the rent profile $U(\theta)$ is everywhere differentiable and satisfies a local incentive condition. This leads to the assumption that the quantity $q(\theta)$ exists and it is a continuous function of θ.

This paper considers the general problem of adverse selection with or without, constraints. Under certain conditions on payoff functions and especially on the type of constraints the optimal contract exists and it is continuous. Such conditions are satisfied for a wide class of regular adverse selection programs. In some interesting cases this condition is not satisfied and it may lead to a discontinuous optimal contract. We apply the optimal control method to characterize the set of assumption which guarantee the existence and continuity of the optimal contract. Out of three approaches to dynamic optimization (calculus of variations, dynamic programming and optimal control) the optimal control leads to the weakest set of assumptions. This allows us to obtain the most general conditions for existence and continuity of the optimal contract.

Literature: In their book Kamien and Schwartz (2012) pointed out on the possibility of discontinuous and bang-bang control in economic problems. They relaxed the continuity requirement on the control. Martimort and Semenov (2008) consider a common agency problem as of Grossman and Helpman (1994) with adverse selection. They showed that the optimal contract is continuous. The optimal set of contracts satisfies certain set of state constraints. This case is a particular case of our results.

Mixed constraints became particularly important in the relational contracts literature. Levin (2003) assumes the continuity of the optimal contract Martimort, Semenov and Stole (20013) consider the enforcement program. They focus on continuous stationary contracts. The results of this paper show that it is without loss of generality; the optimal contract is indeed continuos. Armstrong and Vickers (2011) proved the existence of the optimal contract in a setting without transfers.

Set up

We consider a relationship between a buyer (the principal sometimes referred as *she* in the sequel) and a seller (the agent or *he*) who provides a service or good on her behalf. A contract is a vector function of (non-negative) outputs and payments (q, t). The buyer and the seller have quasi-linear utility functions defined over trade profiles which are respectively given by:

$$V(q,t) = S(q) - t \text{ and } U(q,t,\theta) = t - \theta q$$

where the principal's gross surplus function S is increasing and strictly concave ($S' > 0 > S''$) and satisfies $S(0) = 0$.

We assume that the seller has private information about his cost parameter θ before contracting. The principal only knows the cumulative distribution function $F(\bullet)$ of that cost parameter on an interval support $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$ (with $\Delta\theta > 0$). The the agent knows the realization of the parameter

θ. We assume that the vector (q, t, θ) satisfies a constraint

$$h(q, t, \theta) \geq 0$$

for all admissible q, t and θ.

The Revelation Principle holds; the principal offers a contract to each realization of $\theta : (q(\theta), t(\theta))_{\theta \in \Theta}$. Hence the principal's program $(\mathcal{P})$ is

$$(\mathcal{P}) : \max_{q(\theta), t(\theta)} \int_{\Theta} (S(q(\theta)) - t(\theta)) \, dF(\theta) \text{ such that}$$
$$U(q(\theta), t(\theta), \theta) = t(\theta) - \theta q(\theta) \geq 0,$$
$$\theta \in \arg\max_{\widehat{\theta}} U(q\left(\widehat{\theta}\right), t\left(\widehat{\theta}\right), \theta),$$
$$h(q(\theta), t(\theta), \theta) \geq 0.$$

We introduce a new variable

$$U(\theta) = \max_{\widehat{\theta}} U(q\left(\widehat{\theta}\right), t\left(\widehat{\theta}\right), \theta).$$

The following Lemma describes standard properties of implementable $q(\theta)$ and $U(\theta)$.

Lemma 1 *The output schedule $q(\theta)$ is non-negative, weakly decreasing, and thus almost everywhere differentiable. The rent profile $U(\theta)$ is absolutely continuous, almost everywhere differentiable, convex and satisfies at any point of differentiability:*

$$-q(\theta) \in \partial U(\theta), \tag{1}$$
$$U(\theta) \ \ \textit{convex}, \tag{2}$$
$$U(\overline{\theta}) \geq 0. \tag{3}$$

We first define the set $\mathcal{A}$ of controls that are admissible for optimization problem $(\mathcal{P})$.

Definition 1 *An allocation $(U(\theta), q(\theta))$ is **admissible** if and only if $U(\theta)$ is Fenchel differentiable for all θ, satisfies (1) and (2) and $q(\theta) \geq 0$ is piecewise continuous.*

The problem $(\mathcal{P})$ can be re-stated as

$$(\mathcal{P}) : \quad \max_{(U(\theta), q(\theta)) \in \mathcal{A}} \int_{\underline{\theta}}^{\overline{\theta}} (S(q(\theta)) - \theta q(\theta) - U(\theta)) \, dF(\theta)$$
$$(1), (2), (3) \text{ and}$$
$$h(\theta, q(\theta), U(\theta)) = \delta S(q(\theta)) - \theta q(\theta) - U(\theta) + (1 - \delta) M \geq 0 \quad \forall \theta \in \Theta. \tag{4}$$

The problem $(\mathcal{P})$ is an optimal control problem with the *mixed constraint* (4) which links both the state U and the control q.

Remark 1 *It is also possible to introduce q as a state variable. In the program $(\mathcal{P})$ we should consider the reduced program with omitted (2).*

Main results

Existence of a solution to $(\mathcal{P})$. We use the Filippov-Cesari existence theorem to prove the existence of a solution to an optimal control problem with mixed constraint (Seierstad and Sydsaeter, 1987, Theorem 2, p. 285).

Proposition 1 *A solution to $(\mathcal{P})$ exists in $\mathcal{A}$.*

Proof. Consider the pair $(U(\theta), q(\theta)) = (0, 0)$. Obviously it is an admissible pair. Thus, the set of admissible pairs is non-empty. Fix now U and θ. By concavity of $\delta S(q) - \theta q$ with respect to q the set $\{q \text{ s.t. } h(\theta, q, U) \geq 0\}$ is an interval. Hence, the set

$$N(U, \theta) = \{(S(q) - \theta q - U) f(\theta) + \gamma : \gamma \leq 0, h(\theta, q, U) \geq 0\}$$

is convex. Because of concavity of $\delta S(q) - \theta q$ there exists a constant b such that $U(\theta) \leq b$ for all θ and for all admissible pairs $(U(\theta), q(\theta))$. Indeed because we must have $\delta S(q(\theta)) - \theta q(\theta) \geq U(\theta) - (1-\delta)M$, the right-hand side cannot be unbounded. The last condition in Theorem 2 (Seierstad and Sydsaeter, 1987) is automatically satisfied. Therefore an optimal allocation $(U(\theta), q(\theta))$ exists.[1] ■

The Hamiltonian for the optimization problem $(\mathcal{P}^*)$ is:

$$H(\theta, U, q, \lambda) = (S(q) - \theta q - U) f(\theta) - \lambda q$$

where λ is the co-state variable associated with (1).

The Lagrangian of this problem is:

$$L(\theta, U, q, \lambda, \mu) = H(\theta, U, q, \lambda) + \mu h(\theta, q, U)$$

where μ is the multiplier of the mixed constraint (4).

First, observe that the Hamiltonian and the enforcement constraint (4) are both concave in (U, q) for all θ. Hence, the necessary conditions for optimality of a program with mixed constraint (Seierstad and Sydsaeter, 1987, Theorem 1, p. 276) are also sufficient conditions of the Mangasarian type (Seierstad and Sydsaeter, 1987, Theorem 5, p. 287).

The admissible pair $(U(\theta), q(\theta))$ is optimal if and only if there exist a continuous and piecewise continuously differentiable function $\lambda(\theta)$ and a piecewise continuous function $\mu(\theta)$ such that the following conditions hold:

1. *Optimality with respect to output:*

$$\frac{\partial L}{\partial q}(\theta, U(\theta), q(\theta), \lambda(\theta), \mu(\theta)) = 0 \quad \forall \theta \in \Theta$$

$$\Leftrightarrow f(\theta)(S'(q(\theta))) - \theta) + \mu(\theta)(\delta S'(q(\theta))) - \theta) = \lambda(\theta) \quad \forall \theta \in \Theta. \tag{5}$$

[1] In the Filippov-Cesari Theorem, the optimal control is assumed to be a measurable function. However, the comments following the statement of Seierstad and Sydsaeter (1987, Theorem 8 on p. 133) show that it can be assumed that the optimal control is actually a piecewise continuous function.

2. *Condition on the co-state variable:*

$$\dot{\lambda}(\theta) = -\frac{\partial L}{\partial U}(\theta, U(\theta), q(\theta), \lambda(\theta), \mu(\theta)) \quad \text{for a.e. } \theta \in \Theta$$

$$\Leftrightarrow \dot{\lambda}(\theta) = f(\theta) + \mu(\theta), \quad \text{for a.e. } \theta \in \Theta. \tag{6}$$

3. *Boundary and transversality conditions:*

$$\lambda(\underline{\theta}) = 0 \text{ and } U(\bar{\theta}) = 0. \tag{7}$$

4. *Slackness conditions:*

$$\mu(\theta) \geq 0, h(\theta, q(\theta), U(\theta)) \geq 0 \text{ and } \mu(\theta)h(\theta, q(\theta), U(\theta)) = 0 \quad \forall \theta \in \Theta. \tag{8}$$

Using (6) and (7), we immediately obtain the following expression of the adjoint variable:

$$\lambda(\theta) = F(\theta) + \Psi(\theta) \quad \forall \theta \in \Theta, \tag{9}$$

where $\Psi(\theta) = \int_{\underline{\theta}}^{\theta} \mu(\xi)\, d\xi$.

It is convenient to rewrite condition (5) as:

$$S'(q(\theta))\left(1 + \frac{\delta\mu(\theta)}{f(\theta)}\right) = \theta + \frac{\theta\mu(\theta) + F(\theta) + \Psi(\theta)}{f(\theta)} \quad \forall \theta \in \Theta, \tag{10}$$

We now state the following definition:

Definition 2 *The $n-$partition of the support Θ is the minimal set of n intervals (open or closed) $\{\Theta_1, \Theta_2, ..., \Theta_n\}$ such that $\Theta = \Theta_1 \cup \Theta_2 \cup ... \cup \Theta_n$ and $\mu(\theta) = 0$ for all $\theta \in \Theta_{2k}$ and $\mu(\theta) > 0$ for all $\theta \in \Theta_{2k+1}$. Denote the boundaries of an interval Θ_i as θ_{i-1} and θ_i. Assume that $\theta_0 = \underline{\theta}$ and $\theta_0 \leq \theta_1 \leq \theta_2 \leq ... \leq \theta_n = \bar{\theta}$.*

If Θ_1 is non-empty then the partition starts with an interval where $\mu(\theta) > 0$. If, on the other hand, Θ_1 is empty then the partition starts with an interval Θ_2 where $\mu(\theta) = 0$. Formally, in this case $\Theta_1 = (\underline{\theta}, \underline{\theta}) = \varnothing$, $\Theta_2 = [\theta_1, \theta_2)$, where $\theta_1 = \underline{\theta}$.

Because $\mu(\theta) = 0$ for all $\theta \in \Theta_{2k}$, (10) implies

$$S'(q(\theta)) = S'(q_{\Psi_{2k}}(\theta)) = \theta + \frac{F(\theta) + \Psi_{2k}}{f(\theta)}, \text{ for all } \theta \in \Theta_{2k}, \tag{11}$$

where $\Psi_{2k} = \int_{\underline{\theta}}^{\theta_{2k-1}} \mu(\xi)\, d\xi$ is a constant over Θ_{2k}. We establish now that the output $q(\theta)$ is constant over odd intervals Θ_{2k+1}.

Lemma 2 *For each Θ_{2k+1} there exists a constant q_{2k+1} such that $q(\theta) = q_{2k+1}$ for all $\theta \in \Theta_{2k+1}$.*

Proof. By (8) we have $h(\theta, q(\theta), U(\theta)) = 0$ for all $\theta \in \Theta_{2k+1}$. Differentiating this equation with respect to θ leads to

$$\dot{q}(\theta)\left(\delta S'(q(\theta)) - \theta\right) = 0.$$

Suppose that, for some non-empty interval $\Theta' = [\theta', \theta''] \subset \Theta_{2k+1}$, we have $\delta S'(q(\theta)) - \theta = 0$ for all $\theta \in \Theta'$. Then (10) implies

$$\frac{\theta}{\delta}\left(1 + \frac{\delta\mu(\theta)}{f(\theta)}\right) = \theta + \frac{\theta\mu(\theta) + F(\theta) + \Psi(\theta)}{f(\theta)}$$

for all $\theta \in \Theta'$, which in turn implies

$$\frac{1-\delta}{\delta}\theta = \frac{F(\theta) + \Psi(\theta)}{f(\theta)} \text{ for all } \theta \in \Theta'. \tag{12}$$

By Assumption, we have $\frac{F(\theta)+\Psi(\theta')}{f(\theta)} > \frac{1-\delta}{\delta}\theta$ for all $\theta \in (\theta', \theta'']$ and $\frac{F(\theta)+\Psi(\theta'')}{f(\theta)} < \frac{1-\delta}{\delta}\theta$ for all $\theta \in [\theta', \theta'')$. Thus $\frac{F(\theta)+\Psi(\theta'')}{f(\theta)} < \frac{F(\theta)+\Psi(\theta')}{f(\theta)}$. A contradiction which proves that $\dot{q}(\theta) = 0$ for all $\theta \in \Theta_{2k+1}$. ■

Next important step consists of proving the continuity of output.

Proposition 2 *The optimal output $q(\theta)$ is continuous.*

Proof. Note first that $q(\theta)$ is a continuous function for all $\theta \in int\Theta_i$ for all i. Indeed $q(\theta)$ is a constant for odd i by Lemma 2 or it is given by (11) for even i. Thus, if any discontinuity exists it must be on the boundary of partition sets.

Consider first $\theta_{2k}, k > 0$. Assume that $int\Theta_{2k} = (\theta_{2k-1}, \theta_{2k})$ and $int\Theta_{2k+1} = (\theta_{2k}, \theta_{2k+1})$. Our goal is to prove that $q(\theta_{2k}^{(+)}) = \lim_{\theta\to\theta_{2k}+0} q(\theta) = q(\theta_{2k}^{(-)}) = \lim_{\theta\to\theta_{2k}-0} q(\theta)$. We have

$$h(\theta_{2k}^{(+)}, q(\theta_{2k}^{(+)}), U(\theta_{2k}^{(+)})) = \lim_{\theta\to\theta_{2k}+0} h(\theta, q(\theta), U(\theta)) = 0$$

and

$$h(\theta_{2k}^{(-)}, q(\theta_{2k}^{(-)}), U(\theta_{2k}^{(-)})) = \lim_{\theta\to\theta_{2k}-0} h(\theta, q(\theta), U(\theta)) \geq 0.$$

This implies

$$\delta S(q(\theta_{2k}^{(-)})) - \delta S(q(\theta_{2k}^{(+)})) - \theta_{2k} q(\theta_{2k}^{(-)}) + \theta_{2k} q(\theta_{2k}^{(+)}) \geq 0,$$

which in turn implies

$$\int_{q(\theta_{2k}^{(+)})}^{q(\theta_{2k}^{(-)})} \left(\delta S'(q) - \theta_{2k}\right) dq \geq 0. \tag{13}$$

On the other hand by (10) we have

$$S'(q(\theta_{2k}^{(+)})) - S'(q(\theta_{2k}^{(-)})) = \frac{\mu\left(\theta_{2k}^{(+)}\right)}{f(\theta_{2k})}\left(\theta_{2k} - \delta S'(q(\theta_{2k}^{(+)}))\right). \tag{14}$$

Suppose $q\left(\theta_{2k}^{(+)}\right) > q\left(\theta_{2k}^{(-)}\right)$. Then, $S'(q(\theta_{2k}^{(+)})) < S'(q(\theta_{2k}^{(-)}))$ and by (14) $\theta_{2k} - \delta S'(q(\theta_{2k}^{(+)})) < 0$. Hence

$$\int_{q(\theta_{2k}^{(+)})}^{q(\theta_{2k}^{(-)})} \left(\delta S'(q) - \theta_{2k}\right) dq = -\int_{q(\theta_{2k}^{(-)})}^{q(\theta_{2k}^{(+)})} \left(\delta S'(q) - \theta_{2k}\right) dq =$$

$$\int_{q(\theta_{2k}^{(-)})}^{q(\theta_{2k}^{(+)})} (\theta_{2k} - \delta S'(q))\, dq < \left(\theta_{2k} - \delta S'(q(\theta_{2k}^{(+)}))\right) \left(q(\theta_{2k}^{(+)}) - q(\theta_{2k}^{(-)})\right) < 0,$$

which contradicts (13).

Suppose now that $q\left(\theta_{2k}^{(+)}\right) < q\left(\theta_{2k}^{(-)}\right)$. Then we have by (14) $\theta_{2k} - \delta S'(q(\theta_{2k}^{(+)})) > 0$. Hence

$$\int_{q(\theta_{2k}^{(+)})}^{q(\theta_{2k}^{(-)})} (\delta S'(q) - \theta_{2k})\, dq < \left(\delta S'(q(\theta_{2k}^{(+)})) - \theta_{2k}\right) \left(q(\theta_{2k}^{(-)}) - q(\theta_{2k}^{(+)})\right) < 0,$$

which again contradicts (13). Thus it must be that $q\left(\theta_{2k}^{(+)}\right) = q\left(\theta_{2k}^{(-)}\right)$.

Consider now $\theta_{2k+1}, k \geq 0$. Assume that $\Theta_{2k+1} = (\theta_{2k}, \theta_{2k+1})$ and $\Theta_{2k+2} = (\theta_{2k+1}, \theta_{2k+2})$. We have $h\left(\theta_{2k+1}^{(-)}, ...\right) = 0$ and $h\left(\theta_{2k+1}^{(+)}, ...\right) \geq 0$. This implies

$$\delta S(q(\theta_{2k+1}^{(+)})) - \delta S(q(\theta_{2k+1}^{(-)})) - \theta_{2k+1} q(\theta_{2k+1}^{(+)}) + \theta_{2k+1} q(\theta_{2k+1}^{(-)}) \geq 0,$$

which in turn implies

$$\int_{q(\theta_{2k+1}^{(-)})}^{q(\theta_{2k+1}^{(+)})} (\delta S'(q) - \theta_{2k+1})\, dq \geq 0.$$

On the other hand by (10) we have

$$S(q(\theta_{2k+1}^{(-)})) - S(q(\theta_{2k+1}^{(+)})) = \frac{\mu\left(\theta_{2k+1}^{(-)}\right)}{f(\theta_{2k+1})} \left(\theta_{2k+1} - \delta S(q(\theta_{2k+1}^{(-)}))\right).$$

Suppose $q(\theta_{2k+1}^{(-)}) > q(\theta_{2k+1}^{(+)})$. Then, $S(q(\theta_{2k+1}^{(-)})) < S(q(\theta_{2k+1}^{(+)}))$ and $\theta_{2k+1} < \delta S(q(\theta_{2k+1}^{(-)}))$. Hence

$$\int_{q(\theta_{2k+1}^{(-)})}^{q(\theta_{2k+1}^{(+)})} (\delta S'(q) - \theta_{2k+1})\, dq = -\int_{q(\theta_{2k+1}^{(+)})}^{q(\theta_{2k+1}^{(-)})} (\delta S'(q) - \theta_{2k+1})\, dq =$$

$$\int_{q(\theta_{2k+1}^{(+)})}^{q(\theta_{2k+1}^{(-)})} (\theta_{2k+1} - \delta S'(q))\, dq < \left(\theta_{2k+1} - \delta S(q(\theta_{2k+1}^{(-)}))\right) \left(q(\theta_{2k+1}^{(-)}) - q(\theta_{2k+1}^{(+)})\right) < 0.$$

A contradiction. Suppose now that $q(\theta_{2k+1}^{(-)}) < q(\theta_{2k+1}^{(+)})$. Then (14) yields $\theta_{2k+1} > \delta S(q(\theta_{2k+1}^{(-)}))$. Hence

$$\int_{q(\theta_{2k+1}^{(-)})}^{q(\theta_{2k+1}^{(+)})} (\delta S'(q) - \theta_{2k+1})\, dq < \left(\delta S(q(\theta_{2k+1}^{(-)})) - \theta_{2k+1}\right) \left(q(\theta_{2k+1}^{(+)}) - q(\theta_{2k+1}^{(-)})\right) < 0,$$

which is again a contradiction. Thus it must be that $q(\theta_{2k+1}^{(-)}) = q(\theta_{2k+1}^{(+)})$. ■

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